

Hurewicz theorems for definable groups, Lie groups and their cohomologies

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Abstract

Here we prove an analogue of the Hurewicz theorem for definable groups in an arbitrary o-minimal context, and as an application prove a conjecture of Berarducci on the cohomology of definable groups and Lie groups.

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1 Introduction

Let \mathcal{N} be an arbitrary o-minimal structure. In this paper definable means definable in \mathcal{N} (possibly) with parameters. We will assume that the reader is familiar with the basic theory of o-minimality ([4]) and of definable groups ([5], [14], [16], [18] and [19]).

Given a definable group G we will introduce a kind of fundamental group, $\eta(G)$, the *classifying group* of G , and we prove here the following Hurewicz theorem connecting $\eta(G)$ and the o-minimal Čech cohomology group $\check{H}^1(G; Z)$, from [9]:

Theorem 1.1 *Let G be a definably connected definable group and Z a finite abelian group. Then we have a homomorphism*

$$\mathrm{Hom}_{\mathrm{cont}}(\eta(G), Z) \longrightarrow \check{H}^1(G; Z),$$

where $\mathrm{Hom}_{\mathrm{cont}}(\eta(G), Z)$ is the group of all continuous homomorphisms from the profinite group $\eta(G)$ into Z . Further, this homomorphism does not send any surjective elements of $\mathrm{Hom}_{\mathrm{cont}}(\eta(G), Z)$ to zero.

Now let \mathcal{N} be a sufficiently saturated o-minimal expansion of an ordered field $(N, 0, 1, +, \cdot, <)$. In the paper [3] by Berarducci, Otero, Peterzil and Pillay it is proved that there exists a smallest type definable normal subgroup G^{00} of G such that G/G^{00} with the logic topology is a connected, compact, Lie group. We also have from [10] that the o-minimal singular cohomology of G , $H^*(G; \mathbb{Q})$, is isomorphic to $H^*(G/G^{00}; \mathbb{Q})$, the cohomology of G/G^{00} .

However, in [1] Berarducci points out that, since this theorem was proved in [10] independently of the results on G/G^{00} , the isomorphism

$$H^*(G; \mathbb{Q}) \simeq H^*(G/G^{00}; \mathbb{Q})$$

in the abelian case is purely abstract and is not induced in any obvious way by the quotient homomorphism $\rho_G : G \longrightarrow G/G^{00}$. See [1] Remark 7.6.

As an application of our Hurewicz result, and the construction used to prove it we show how it can be seen that the quotient homomorphism $\rho_G : G \longrightarrow G/G^{00}$ induces the isomorphism $H^*(G; \mathbb{Q}) \simeq H^*(G/G^{00}; \mathbb{Q})$ in cohomology. We will do this in the case of definably compact abelian groups and in the case of definably semi-simple groups, obtaining, for example, the following:

Theorem 1.2 *Assume that G is a definably compact, definable abelian group. Then there exists an isomorphism*

$$H^*(G; \mathbb{Q}) \simeq H^*(G/G^{00}; \mathbb{Q})$$

induced by the quotient homomorphism $\rho_G : G \longrightarrow G/G^{00}$.

2 The Hurewicz theorem

Throughout this section G is a definably connected definable group. Here we will prove the Hurewicz theorem for G . In the first three subsections we introduce the classifying group, $\eta(G)$ and the two groups of equivalence classes of definable Z -covering homomorphisms of G , namely $Z\text{-Cov}(G)$ and $\text{Ext}_{\text{def}}(G; Z)$. We also note that we have a surjective homomorphism $P : \text{Ext}_{\text{def}}(G; Z) \longrightarrow Z\text{-Cov}(G)$. We then proceed to define a monomorphism

$$A : Z\text{-Cov}(G) \longrightarrow \check{H}^1(G; Z),$$

a map

$$B : \text{Hom}_{\text{cont}}(\eta(G), Z) \longrightarrow \text{Ext}_{\text{def}}(G; Z)$$

and an isomorphism

$$C : \text{Ext}_{\text{def}}(G; Z) \longrightarrow \text{Hom}_{\text{cont}}(\eta(G), Z).$$

Since we also show that B is the inverse of C , it follows that B is an isomorphism and the homomorphism

$$A \circ P \circ B : \text{Hom}_{\text{cont}}(\eta(G), Z) \longrightarrow \check{H}^1(G; Z)$$

proves Theorem 1.1.

2.1 The classifying group $\eta(G)$

We introduce here another notion of "fundamental group" for o-minimal structures which will classify the definable covering homomorphisms $h : H \longrightarrow G$ of a definably connected definable group G just as $\pi(G)$ from [6] classifies the locally definable covering homomorphisms $h : H \longrightarrow G$ of a definably connected locally definable group G .

Consider the category $\text{Cov}_{\text{def}}(G)$ whose objects are the continuous surjective definable homomorphisms $h : H \longrightarrow G$, where H is a definable group, such that there is a definable cover $\{U_i\}_{i=1}^k$ of G such that each $h^{-1}(U_i)$ is a disjoint union of open definable subsets of H , each of which is mapped homeomorphically by h onto U_i . By [6] Theorem 3.6 and o-minimality these are precisely the definable surjective homomorphisms $h : H \longrightarrow G$ with finite kernel. A morphism between two objects $h : H \longrightarrow G$ and $k : K \longrightarrow G$ in $\text{Cov}_{\text{def}}(G)$ is a definable covering homomorphism $l : H \longrightarrow K$ such that $h = k \circ l$ (by [6] Theorem 3.6, this means that l is a surjective definable homomorphism with finite kernel). Also consider the full subcategory $\text{Cov}_{\text{def}}^0(G)$ of $\text{Cov}_{\text{def}}(G)$ of definable homomorphisms $h : H \longrightarrow G$ where H is definably connected. Following the arguments in [6], we see that $\text{Cov}_{\text{def}}(G)$ and $\text{Cov}_{\text{def}}^0(G)$ are inverse systems. Thus we can make the following

Definition 2.1 *The kernel of the inverse limit $\tilde{p}_{\text{def}} : \tilde{G}_{\text{def}} \rightarrow G$ of the inverse system $\text{Cov}_{\text{def}}^0(G)$ will be called the classifying group of G and is denoted by $\eta(G)$.*

We have the useful characterization of $\eta(G)$ as the inverse limit of the inverse system of groups $\text{Aut}(K/G)$ (or $\text{Ker}(h)$) of definable covering automorphisms, directed by the homomorphisms $\theta_H^K : \text{Aut}(K/G) \rightarrow \text{Aut}(H/G)$ (or $l_H^K : \text{Ker}(k) \rightarrow \text{Ker}(h)$) induced by (or obtained by restriction from) $K \rightarrow H \in \text{MorCov}_{\text{def}}^0(G)$.

The argument used in the proof of Theorem 1.1 from [6], carries to the category $\text{Cov}_{\text{def}}^0(G)$ and gives:

Theorem 2.2 *Let G be a definably connected definable group. Then $\eta(G)$ is abelian. Moreover, if G is abelian, then the following hold:*

- (1) *the group \tilde{G}_{def} is divisible and torsion free;*
- (2) *the classifying group $\eta(G)$ of G is torsion-free;*
- (3) *the k -torsion subgroup of G is isomorphic to $\eta(G)/k\eta(G)$, for each $k > 0$.*

Note that in Theorem 2.2 we used also the fact that if G is a definably connected, definable abelian group then, for each $k > 0$, the subgroup of k -torsion points of G has dimension zero (i.e. is finite) by [21] and hence G is k -divisible since by definably connectedness it has no proper definable subgroups of finite index ([18]).

In fact, $\eta(G)$ may well have more nice properties than originally expected, for example:

Theorem 2.3 *Assume that \mathcal{N} is an o-minimal expansion of a group, and let G be a definably connected definable group. Then $\eta(G)$ is the profinite completion of $\pi_1(G)$, the o-minimal fundamental group of G (denoted $\eta(G) = \widehat{\pi_1(G)}$).*

Proof. By [7] Theorem 1.4 $\pi_1(G)$ is isomorphic to the group $\pi(G)$ which is the kernel of the inverse limit $\tilde{p} : \tilde{G} \rightarrow G$ of the inverse system $\text{Cov}^0(G)$ of definably connected *locally* definable covering homomorphisms of G , with morphisms being also *locally* definable covering homomorphisms. i.e. in this context $\pi_1(G) \simeq \varprojlim (\text{Aut}(H/G))$, where the inverse limit ranges over the category $\text{Cov}^0(G)$.

By [7] Theorem 1.4 again, we have that for every subgroup $L \leq \pi_1(G)$ there exists $h : H \rightarrow G \in \text{Cov}^0(G)$ such that $\pi_1(G)/L \simeq \text{Aut}(H/G) \simeq \text{ker}h$. Indeed one can take $h : H \rightarrow G$ to be the locally definable homomorphism $\tilde{G}/L \rightarrow G$ induced by the o-minimal universal covering homomorphism $\tilde{p} : \tilde{G} \rightarrow G$ which is a locally definable covering homomorphism.

Now observe that the collection of finite index subgroups of $\pi_1(G)$ will form a filter base: suppose that $L_1, L_2 \leq \pi_1(G)$ and $L = L_1 \cap L_2$, then by the above, there exists $p : H \rightarrow G \in \text{Cov}^0(G)$ such that $\pi_1(G)/L \simeq \text{Aut}(H/G)$. But the injective homomorphism $\pi_1(G)/L \rightarrow \pi_1(G)/L_1 \times \pi_1(G)/L_2 : a + L \mapsto (a + L_1, a + L_2)$ shows that $p : H \rightarrow G \in \text{Cov}_{\text{def}}^0(G)$ (since $\text{Aut}(H/G) \simeq \text{Ker}(p)$ will then be finite). \square

So we also make the following, more general

Conjecture 2.4 *For every definably connected definable group G , we have that $\eta(G)$ is the profinite completion of $\pi(G)$ (i.e. $\eta(G) = \widehat{\pi(G)}$).*

In the case that G is abelian we have that $\eta(G)$ is a torsion-free profinite abelian group (Theorem 2.2), so by [20] Theorem 4.3.3 we have $\eta(G) = \prod_p \text{prime} \Pi_{\kappa(p)} \mathbb{Z}_p$ where for each prime p , \mathbb{Z}_p is the group of p -adic integers and $\kappa(p)$ is some cardinal number. So it is natural to make the following

Conjecture 2.5 *If G is a definably connected definably compact definable abelian group, then $\eta(G) \simeq \widehat{\mathbb{Z}}^{\dim G}$ and in particular, $G[k] \simeq (\mathbb{Z}/k\mathbb{Z})^{\dim G}$ (by Theorem 2.2).*

2.2 The group $Z\text{-Cov}(G)$

We recall from Subsection 2.1 that $\text{Cov}_{\text{def}}(G)$ is the collection of definable surjective homomorphisms $p_H : H \rightarrow G$ such that there is a finite open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of G by definable open sets with the property that for each $\alpha \in I$, $p_H^{-1}(U_\alpha) = U_\alpha^1 \dot{\cup} \dots \dot{\cup} U_\alpha^m$ is a finite disjoint union of definable open sets each of which is sent homeomorphically to U_α by the definable map $p_H|_{U_\alpha^i}$, for $i = 1, \dots, m$ (note here that $m = |\text{Ker}(p_H)|$). We say that p_H is *trivial* over such a \mathcal{U} .

Definition 2.6 Given a finite abelian group Z , a *definable Z -covering homomorphism* of a definable group G is a definable covering homomorphism $p_H : H \rightarrow G$ of G such that $\text{Ker}(p_H) = Z$.

Equivalently a definable Z -covering homomorphism is a group extension, H , of Z with a definable covering homomorphism $p_H : H \rightarrow G$ such that the following sequence is exact:

$$e_Z \hookrightarrow Z \hookrightarrow H \twoheadrightarrow_{p_H} G \twoheadrightarrow e_G.$$

Two definable covering homomorphisms, p_H and p_K are *definably homeomorphic* if there is a definable homeomorphism $F : H \rightarrow K$ such that the

following diagram commutes:

$$\begin{array}{ccccc}
 & & H & & \\
 & & \uparrow & & \\
 e_Z & \longrightarrow & Z & \xrightarrow{F} & G & \longrightarrow & e_G \\
 & & \downarrow & & \uparrow & & \\
 & & K & & & &
 \end{array}$$

(i.e. $p_H = p_K \circ F$ and $F|_Z = \text{id}_Z$).

We denote the set of definable Z -covering homomorphisms of G , up to definable homeomorphism, by $Z\text{-Cov}_{\text{def}}(G)$, and the subset which are trivial over a suitable cover \mathcal{U} by $Z\text{-Cov}_{\text{def}}(G; \mathcal{U})$.

Having $\text{Ker}(p_H) = Z$ gives us a definable action of Z on H , by right multiplication, and clearly G is the result of factoring by this action. When viewing Z in this way we write x^z for the result of the element $z \in Z$ acting on $x \in G$.

As $p_H(x^z) = p_H(x.z) = p_H(x).p_H(z) = p_H(x)$ for $z \in \text{Ker}(p_H)$ we see that the action of Z on H simply permutes the fibres of p_H .

Lemma 2.7 *For any (H, p_H) and (K, p_K) definable Z -covering homomorphisms of G which are definably homeomorphic via F we have, for any $x \in H$ and $z \in Z$ that $F(x^z) = F(x)^z$.*

Proof. We will prove this for $x \in H^0$, the definably connected component of H , and then note that since the restricted map $p_H : H^0 \rightarrow G$ is surjective (by Corollary 3.7 in [6]) we have that for any $x \in H$ there is $x' \in H^0$ such that $p_H(x) = p_H(x')$, so $x^{-1}.x' \in \text{Ker}(p_H) = Z$, so there is $z' \in Z$ such that $x = x'^{z'}$. Then, using the result for x' we get

$$F(x^z) = F((x'^{z'})^z) = F(x'^{z'.z}) = F(x')^{z'.z} = (F(x')^{z'})^z = F(x)^z.$$

So fix z for the moment and take $x \in H^0$. Note that we have, from the commutative diagram in the definition, that $p_K(F(x^z)) = p_H(x^z) = p_H(x) = p_K(F(x))$. This gives us that $F(x^z).(F(x))^{-1} \in \text{Ker}(p_K) = Z$, and so we can define a definable continuous function $h : H^0 \rightarrow Z$ by $h(x) = F(x^z).(F(x))^{-1}$. But H^0 is definably connected and Z is discrete, so this function is constant, say $= z_0$, so $F(x^z) = F(x).z_0$. But letting $x = e_H$ we see that we must have $z_0 = z$, so that $F(x^z) = F(x).z = F(x)^z$. \square

Lemma 2.8 $Z\text{-Cov}_{\text{def}}(G; \mathcal{U})$, and thus $Z\text{-Cov}_{\text{def}}(G)$ also, is a group.

Proof. Given $p_H: H \rightarrow G$ and $p_K: K \rightarrow G$ in $Z\text{-Cov}_{\text{def}}(G; \mathcal{U})$ we define the product

$$H * K = \frac{H \times_G K}{\Delta_Z^*},$$

and the map

$$p_H * p_K: H * K \rightarrow G$$

to be given by $p_H * p_K([(x, y)]) = p_H(x) = p_K(y)$.

Here $H \times_G K = \{(x, y) \in H \times K \mid p_H(x) = p_K(y)\}$ is the fibre product of H and K over G , and $\Delta_Z^* = \{(z, -z) \mid z \in Z\}$, so the map is well defined.

It is a definable Z -covering homomorphism since it is clearly definable and surjective, and

$$\text{Ker}(p_H * p_K) = \frac{\text{Ker}(p_H) \times \text{Ker}(p_K)}{\Delta_Z^*} = \frac{Z \times Z}{\Delta_Z^*} = Z.$$

(the last equality is justified as we can identify $\frac{Z \times Z}{\Delta_Z^*}$ with Z using addition,

since the sequence $0 \rightarrow \Delta_Z^* \hookrightarrow Z \times Z \xrightarrow{+} Z \rightarrow 0$ is exact). This covering is trivial over \mathcal{U} since, if V_1, \dots, V_k are the open definable sets in K above $U \in \mathcal{U}$ and W_1, \dots, W_h are the open definable sets in H above the same U , then the sets $(W_i \times_G V_j) / \Delta_Z^*$ are in $H * K$ above U , and by the definition of the fibre product are definably homeomorphic to U .

It is also clear that any two pairs of definable Z -covering homomorphisms which are definably homeomorphic give definably homeomorphic products.

This turns $Z\text{-Cov}_{\text{def}}(G; \mathcal{U})$ into a group whose identity is the trivial definable Z -covering $G \times Z \rightarrow G$, and where the inverse of $p_H: H \rightarrow G$ is the definable Z -covering homomorphism $p_H^*: H^* \rightarrow G$, where $H^* = \{x^{-1} \mid x \in H\}$ has product $x^{-1} \cdot y^{-1} = (x \cdot y)^{-1}$ and $p_H^*(x^{-1}) = p_H(x)$. \square

2.3 The group $\text{Ext}_{\text{def}}(G; Z)$

We give here a definition of the group $\text{Ext}_{\text{def}}(G; Z)$ from [5] which is equivalent (in this context) to the one given there. We also make one simple observation about this group.

The set $\text{Ext}_{\text{def}}(G; Z)$ is simply the collection of all equivalence classes of definable Z -covering homomorphisms under the equivalence relation of *definable equivalence*. We say that $p_H: H \rightarrow G$ and $p_K: K \rightarrow G$ are definably equivalent if there is a definable isomorphism of groups $\phi: H \rightarrow K$ such that $p_H = p_K \circ \phi$ and $\phi|_Z = \text{id}_Z$. We define a binary operation, $*$, on $\text{Ext}_{\text{def}}(G; Z)$ in exactly the same way as for $Z\text{-Cov}_{\text{def}}(G)$, and it is still clearly a well-defined group operation.

The set $Z\text{-Cov}_{\text{def}}(G)$ is just the collection of all equivalence classes of definable Z -covering homomorphisms under the equivalence relation of definable homeomorphism of Z -covering homomorphisms. Every map which

demonstrates a definable equivalence is a definable isomorphism, and so is also a definable homeomorphism. Thus the equivalence relation of definable homeomorphism is coarser than definable equivalence, so we have that $Z\text{-Cov}_{\text{def}}(G) = \text{Ext}_{\text{def}}(G; Z) / \sim$, where \sim is definable homeomorphism of Z -covering homomorphisms. Thus we have a (projection) homomorphism, P , from $\text{Ext}_{\text{def}}(G; Z)$ to $Z\text{-Cov}_{\text{def}}(G)$.

2.4 The monomorphism $A: Z\text{-Cov}_{\text{def}}(G) \hookrightarrow \check{H}^1(G, Z)$.

We first adapt an argument from [12] to show that this map exists.

Lemma 2.9 *For any finite cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of G by definable open sets there is a well-defined map:*

$$A_{\mathcal{U}}: Z\text{-Cov}_{\text{def}}(G; \mathcal{U}) \hookrightarrow \check{H}^1(G, \mathcal{U}, Z).$$

Proof. Let $p_H: H \rightarrow G$ be a definable Z -covering homomorphism which is trivial over $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. We define definable homeomorphisms

$$\phi_\alpha^i: (U_\alpha \times Z) \rightarrow p_H^{-1}(U_\alpha)$$

for each $\alpha \in I$ and $i = 1, \dots, m$ (where $p_H^{-1}(U_\alpha) = U_\alpha^1 \dot{\cup} \dots \dot{\cup} U_\alpha^m$). For each $u \in U_\alpha$ and $i = 1, \dots, m$ there is a unique $u_i \in U_\alpha^i$ such that $p_H(u_i) = u$. Since Z acts on H , and thus on $p_H^{-1}(U_\alpha)$ we can then define

$$\phi_\alpha^i(u, z) := u_i^z,$$

and this is a definable map (and clearly a homeomorphism, as the copies of U_α are simply rearranged).

Then, for $x \in p_H^{-1}(U_\alpha)$ we have that $(\phi_\alpha^i)^{-1}(x) = (p_H(x), z_x)$ where z_x is the unique element of Z such that $x = ((p_H(x))_i)^{z_x}$ (this z_x exists and is unique since $p_H((p_H(x))_i) = p_H(x)$ so that $(p_H(x))_i \cdot x^{-1} \in \text{Ker}(p_H) = Z$.)

So the definable composition:

$$(U_\alpha \cap U_\beta) \times Z \xrightarrow{\phi_\alpha^i} p_H^{-1}(U_\alpha \cap U_\beta) \xrightarrow{(\phi_\beta^j)^{-1}} (U_\alpha \cap U_\beta) \times Z$$

is given by

$$(\phi_\beta^j)^{-1} \circ \phi_\alpha^i(u, z) = (\phi_\beta^j)^{-1}(u_i^z) = (p_H(u_i^z), z') = (u, z'),$$

where z' is the unique element of Z such that $u_i^z = u_j^{z'}$. For $u \in U_\alpha \cap U_\beta$ we know that $u_i = u_j^{z_{\alpha, \beta}^{i, j}}$ for some unique $z_{\alpha, \beta}^{i, j} \in Z$ (again since $u_i \cdot u_j^{-1} \in \text{Ker}(p_H) = Z$). Thus $u_i^z = (u_j^{z_{\alpha, \beta}^{i, j}})^z = u_j^{z_{\alpha, \beta}^{i, j} + z}$, and so we have $z' = z_{\alpha, \beta}^{i, j} + z$ and

$$(\phi_\beta^j)^{-1} \circ \phi_\alpha^i(u, z) = (u, z + z_{\alpha, \beta}^{i, j}).$$

Now we fix a choice of homeomorphisms $\{\phi_\alpha^{i_\alpha}\}_{\alpha \in I}$ so that for each ordered pair $\langle \alpha, \beta \rangle \in I^2$ such that $U_\alpha \cap U_\beta$ is non-empty we have a fixed element $z_{\alpha, \beta}^{i_\alpha, i_\beta} \in Z$, i.e. associated to the definable Z -covering homomorphism $p_H : H \rightarrow G$ we have a map $c : \{\langle \alpha, \beta \rangle \in I^2 | U_\alpha \cap U_\beta \neq \emptyset\} \rightarrow Z$ where $c(\alpha, \beta) = z_{\alpha, \beta}^{i_\alpha, i_\beta}$. This c is an element of the Čech group $\check{C}^1(G, \mathcal{U}; Z)$.

To show that the $c \in \check{C}^1(G, \mathcal{U}; Z)$ given by $\{\phi_\alpha^{i_\alpha}\}_{\alpha \in I}$ is not just a co-chain, but in fact determines an element of the cohomology group we must also show that $c \in \text{Ker}(\delta_1)$ (i.e. that c determines a co-cycle). This is the case iff for any $\langle \alpha, \beta, \gamma \rangle \in I^3$ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ we have that $(\delta_1 c)(\alpha, \beta, \gamma) = -c(\beta, \gamma) + c(\alpha, \gamma) - c(\alpha, \beta) = 0$. But as $(\phi_\gamma^{i_\gamma})^{-1} \circ \phi_\alpha^{i_\alpha} = ((\phi_\gamma^{i_\gamma})^{-1} \circ \phi_\beta^{i_\beta}) \circ ((\phi_\beta^{i_\beta})^{-1} \circ \phi_\alpha^{i_\alpha})$ this follows, as then for any $z \in Z$ we have

$$\begin{aligned} (u, z + z_{\alpha, \gamma}) &= (\phi_\gamma^{i_\gamma})^{-1} \circ \phi_\alpha^{i_\alpha}(u, z) \\ &= ((\phi_\gamma^{i_\gamma})^{-1} \circ \phi_\beta^{i_\beta}) \circ ((\phi_\beta^{i_\beta})^{-1} \circ \phi_\alpha^{i_\alpha})(u, z) = (u, z + z_{\alpha, \beta} + z_{\beta, \gamma}), \end{aligned}$$

which then gives $-z_{\alpha, \beta} + z_{\alpha, \gamma} - z_{\beta, \gamma} = 0$ as required.

We now have to show that our choice of homeomorphisms $\{\phi_\alpha^{i_\alpha}\}_{\alpha \in I}$ is factored out when we pass from the Čech group to the cohomology groups. That is, we need to show that if $\{\phi_\alpha^{i'_\alpha}\}_{\alpha \in I}$ is another choice from our set of homeomorphisms and is such that $(\phi_\beta^{i'_\beta})^{-1} \circ \phi_\alpha^{i'_\alpha}(u, z) = (u, z + z_{\alpha, \beta}^{i'_\alpha, i'_\beta})$ ($z_{\alpha, \beta}^{i'_\alpha, i'_\beta}$ a constant in Z) and we define the map $c' : \{\langle \alpha, \beta \rangle \in I^2 | U_\alpha \cap U_\beta \neq \emptyset\} \rightarrow Z$ by $c'(\alpha, \beta) = z_{\alpha, \beta}^{i'_\alpha, i'_\beta}$ then c and c' determine the same equivalence class modulo $\text{Im}\delta_0$, i.e. $(c' - c) \in \text{Im}\delta_0$, i.e. there is $(h : I \rightarrow Z) \in \check{C}^0(\mathcal{U}; Z)$ such that for all $\langle \alpha, \beta \rangle$ we have $c'(\alpha, \beta) = (\delta_0(h))(\alpha, \beta) + c(\alpha, \beta) = h(\alpha) + c(\alpha, \beta) - h(\beta)$.

If $\{\phi_\alpha^{i'_\alpha}\}_{\alpha \in I}$ is as described then, by the above, for each $\alpha \in I$ we have a homeomorphism $(\phi_\alpha^{i'_\alpha})^{-1} \circ \phi_\alpha^{i'_\alpha}$ of $U_\alpha \times Z$ to itself given by

$$(\phi_\alpha^{i'_\alpha})^{-1} \circ \phi_\alpha^{i'_\alpha}(u, z) = (u, z + x_\alpha),$$

where $x_\alpha = z_{\alpha, \alpha}^{i'_\alpha, i'_\alpha}$. Clearly this map is such that the following diagram commutes:

$$\begin{array}{ccc} U_\alpha \times Z & \xrightarrow{(u, z) \mapsto (u, z + x_\alpha)} & U_\alpha \times Z \\ & \searrow \phi_\alpha^{i'_\alpha} & \swarrow \phi_\alpha^{i_\alpha} \\ & & p^{-1}(U_\alpha) \end{array}$$

Thus we have that $\phi_\alpha^{i'_\alpha}(u, z) = \phi_\alpha^{i_\alpha}(u, z + x_\alpha)$ and $\phi_\beta^{i'_\beta}(u, z) = \phi_\beta^{i_\beta}(u, z - x_\beta)$, so that for all $\alpha, \beta \in I$, $u \in U_\alpha \cap U_\beta$ and $z \in Z$ we have:

$$\begin{aligned}
(u, z + z'_{\alpha, \beta}) &= (\phi_\beta^{i'_\beta})^{-1} \circ \phi_\alpha^{i'_\alpha}(u, z) \\
&= (\phi_\beta^{i'_\beta})^{-1} \circ \phi_\alpha^{i_\alpha}(u, z + x_\alpha) \\
&= (\phi_\beta^{i'_\beta})^{-1} \circ \phi_\beta^{i_\beta} \circ ((\phi_\beta^{i_\beta})^{-1} \circ \phi_\alpha^{i_\alpha})(u, z + x_\alpha) \\
&= (\phi_\beta^{i'_\beta})^{-1} \circ \phi_\beta^{i_\beta}(u, z + x_\alpha + z_{\alpha, \beta}) \\
&= (\phi_\beta^{i'_\beta})^{-1} \circ \phi_\beta^{i'_\beta}(u, z + x_\alpha + z_{\alpha, \beta} - x_\beta) \\
&= (u, z + x_\alpha + z_{\alpha, \beta} - x_\beta)
\end{aligned}$$

So $z'_{\alpha, \beta} = x_\alpha + z_{\alpha, \beta} - x_\beta$. As the x_α are constant we can define a map $h: I \rightarrow Z$ such that $h(\alpha) = x_\alpha$ and this is the function we need, since now $c'(\alpha, \beta) = z'_{\alpha, \beta} = x_\alpha + z_{\alpha, \beta} - x_\beta = h(\alpha) + c(\alpha, \beta) - h(\beta)$.

To complete the proof that $A_{\mathcal{U}}$ is well-defined we need to show that if there is a definable homeomorphism of Z -covering homomorphisms $F: (H, p_H) \rightarrow (K, p_K)$, then $[c_H] = [c_K]$. But recall that we have just shown that the choice of homeomorphisms $\{\phi_\alpha^{i_\alpha}\}_{\alpha \in I}$ between $U_\alpha \times Z$ and $p_H^{-1}(U_\alpha)$ from which we obtain c_H is factored out when we go to the equivalence class $[c_H]$. Thus it suffices to show that, for any such choice, we have that $F \circ \phi_\alpha^{i_\alpha} = \psi_\alpha^{j_\alpha}$, for some choice $\{\psi_\alpha^{j_\alpha}\}_{\alpha \in I}$ of such homeomorphisms between $U_\alpha \times Z$ and $p_K^{-1}(U_\alpha)$. Then we get:

$$(\phi_\beta^{i_\beta})^{-1} \circ \phi_\alpha^{i_\alpha} = (F \circ \phi_\beta^{i_\beta})^{-1} \circ (F \circ \phi_\alpha^{i_\alpha}) = (\psi_\beta^{j_\beta})^{-1} \circ \psi_\alpha^{j_\alpha},$$

so using these maps do determine c_H and c_K we get $c_H = c_K$.

So we want to be able to take the j_α to be such that $F(u_{i_\alpha}^z) = u_{j_\alpha}^z$ for all $z \in Z$ and $\alpha \in I$. This is easy since $F(x^z) = F(x)^z$, and so we can just take the j_α to be such that $F(u_{i_\alpha}) = u_{j_\alpha}$, which is possible since $p_H = p_K \circ F$, so $F(u_{i_\alpha}) \in (p_K)^{-1}(p_H(u_{i_\alpha})) = (p_K)^{-1}(u) = \{u_1, \dots, u_m\}$. Thus we have our well defined map, $A_{\mathcal{U}}$, from $Z\text{-Cov}_{\text{def}}(G; \mathcal{U})$ to $\check{H}^1(G, \mathcal{U}, Z)$. \square

Corollary 2.10 *There is a well-defined map:*

$$A: Z\text{-Cov}_{\text{def}}(G) \hookrightarrow \check{H}^1(G, Z).$$

Proof. We only need to note that for any two $p_H: H \rightarrow G$ and $p_K: K \rightarrow G$ in $Z\text{-Cov}_{\text{def}}(G)$ the set

$$\{\mathcal{V} | (p_H: H \rightarrow G), (p_K: K \rightarrow G) \in Z\text{-Cov}_{\text{def}}(G; \mathcal{V})\}$$

is cofinal in the set of all finite open covers of G by definable sets, ordered by refinement. This means that we can take the limit with respect to these covers and get the required map. \square

Lemma 2.11 *A is injective.*

Proof. We show that $A_{\mathcal{U}}$ is injective for each finite open cover of G by definable sets, \mathcal{U} . Then going to the limit as in the proof of the corollary above gives the result.

We suppose that $A_{\mathcal{U}}(p_H) = [c_H]$ where we obtain the function c_H from the (definable) homeomorphisms $\{\phi_{\alpha}^{i_{\alpha}}\}_{\alpha \in I}$, and that $[c_H] = [0]$. We will show that there is another choice $\{\phi_{\alpha}^{i'_{\alpha}}\}_{\alpha \in I}$ of homeomorphisms from $U_{\alpha} \times Z$ to $p_H^{-1}(U_{\alpha})$ such that for any $\alpha, \beta \in I$ and $u \in U_{\alpha} \cap U_{\beta}$ we have $(\phi_{\beta}^{i'_{\beta}})^{-1} \circ \phi_{\alpha}^{i'_{\alpha}}(u, z) = (u, z)$ so that the function, c'_H , obtained from these homeomorphism is the zero function. Once we have done this we have that $\phi_{\alpha}^{i'_{\alpha}} = \phi_{\beta}^{i'_{\beta}}$ on any $(U_{\alpha} \cap U_{\beta}) \times Z$. Thus we can define a definable homeomorphism $F' : G \times Z \rightarrow H$ such that $\text{pr} = p_H \circ F'$ by $F'(u, z) = \phi_{\alpha}^{i'_{\alpha}}(u, z)$ for any α such that $u \in U_{\alpha}$ (where $\text{pr} : G \times Z \rightarrow G$ is the standard projection). In order to get a definable homeomorphism of definable Z -covering homomorphisms we also need to take $F'(e_G, z) = (e_G)_{i_{\alpha}}^z = (e_G)_{i_{\alpha}} \cdot z$ (where $e_G \in U_{\alpha}$), and let $F(u, z) = F'(u, z)^{z_0}$ where $z_0 = -(e_G)_{i_{\alpha}}$. Then we get the commutative diagram:

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow & & \searrow & \\
 G \times Z & \xrightarrow{F'} & H & \xrightarrow{z_0} & H \\
 & \searrow & \downarrow p_H & \swarrow p_H & \\
 & & G & &
 \end{array}$$

F has the same properties as F' (i.e. it is a definable homeomorphism which commutes with the definable covering homomorphisms, since the action of z_0 just permutes their fibres) plus we also get $F(e_G, z) = ((e_G)_{i_{\alpha}}^z)^{-(e_G)_{i_{\alpha}}} = ((e_G)_{i_{\alpha}}^{-(e_G)_{i_{\alpha}}})^z = z$, so that $F|_Z = \text{id}_Z$. This will finish the proof, as the equivalence class of the trivial Z -covering homomorphism is the zero of the group.

But suppose we do have c_H as above. So $c(\alpha, \beta) = z_{\alpha, \beta}$ for the unique $z_{\alpha, \beta}$ such that $u_{i_{\alpha}} = u_{i_{\beta}}^{z_{\alpha, \beta}}$ for any $u \in U_{\alpha} \cap U_{\beta}$, and also there is a function $h : I \rightarrow Z$ such that $c(\alpha, \beta) = h(\beta) - h(\alpha)$ for all α, β . We then have $u_{i_{\alpha}}^{h(\alpha)} = u_{i_{\beta}}^{h(\beta)}$. We let $i'_{\alpha} \in \{1, \dots, m\}$ be the unique choice such that $u_{i_{\alpha}}^{h(\alpha)} = u_{i'_{\alpha}}$. Thus we have, for $u \in U_{\alpha} \cap U_{\beta}$, that $u_{i'_{\alpha}} = u_{i'_{\beta}}$. Now we can

see that the collection $\{\phi'_\alpha\}_{\alpha \in I}$ are clearly the homomorphisms we want as they give $c'_H(\alpha, \beta) = z'_{\alpha, \beta}$ where $z'_{\alpha, \beta}$ is the unique element of Z such that $u_{i'_\alpha} = (u_{i'_\beta})^{z'_{\alpha, \beta}}$. But this $z'_{\alpha, \beta}$ is clearly zero (the identity, e_Z) for all α, β . \square

We note that we cannot follow [12] to get an inverse map from $\check{H}^1(G; Z)$ to $Z\text{-Cov}_{\text{def}}(G)$ as the Z -covering constructed there cannot be shown to be a definable group, let alone the map to be a definable covering homomorphism.

Lemma 2.12 *A is a homomorphism.*

Proof. Again we prove the result for $A_{\mathcal{U}}$, and note that still have a homomorphism when we go to the limit.

Let $p_H: H \rightarrow G$ and $p_K: K \rightarrow G$ be in $Z\text{-Cov}_{\text{def}}(G; \mathcal{U})$, so that we can choose two sets of definable homeomorphisms:

$$\varphi_\alpha: (U_\alpha \times Z) \rightarrow p_H^{-1}(U_\alpha) \text{ and } \psi_\alpha: (U_\alpha \times Z) \rightarrow p_K^{-1}(U_\alpha),$$

for $\alpha \in I$, and then get two sets of definable homeomorphisms of $(U_\alpha \cap U_\beta) \times Z$, namely $(\varphi_\beta^{-1} \circ \varphi_\alpha)$ and $(\psi_\beta^{-1} \circ \psi_\alpha)$ which are, by the argument above, given by, say

$$\varphi_\beta^{-1} \circ \varphi_\alpha(u, z) = (u, z + z_{\alpha, \beta}) \text{ and } \psi_\beta^{-1} \circ \psi_\alpha(u, z) = (u, z + g_{\alpha, \beta}).$$

Then $A_{\mathcal{U}}(p_H) = [c_H]$ and $A_{\mathcal{U}}(p_K) = [c_K]$ where $c_H(\alpha, \beta) = z_{\alpha, \beta}$ and $c_K(\alpha, \beta) = g_{\alpha, \beta}$.

Since $(p_H * p_K)$ is trivial over \mathcal{U} we can let $\mu_\alpha: (U_\alpha \times Z) \rightarrow (p_H * p_K)^{-1}(U_\alpha)$ be some choice of homeomorphism as constructed above and have, as above, that there are $l_{\alpha, \beta} \in Z$ such that $\mu_\beta^{-1} \circ \mu_\alpha(u, z) = (u, z + l_{\alpha, \beta})$ for $U_\alpha \cap U_\beta \neq \emptyset$. Then we have $c_{H*K}(\alpha, \beta) = l_{\alpha, \beta}$ (i.e. $A_{\mathcal{U}}(p_{H*K}) = [c_{H*K}]$) and we want to show that $[c_H] + [c_K] = [c_{H*K}]$. This will certainly follow if we show that, for some choice of homeomorphisms μ_α , we get $l_{\alpha, \beta} = z_{\alpha, \beta} + g_{\alpha, \beta}$.

We can choose the definable homeomorphisms $\mu_\alpha: (U_\alpha \times Z) \rightarrow (p_H * p_K)^{-1}(U_\alpha)$ to be given by:

$$\mu_\alpha(u, z) = \frac{(\varphi_\alpha(u, z), \psi_\alpha(u, z))}{\Delta_Z^*},$$

which is well-defined as the following diagram commutes:

$$\begin{array}{ccc} U_\alpha \times (Z \times Z) & \xrightarrow{(\varphi_\alpha, \psi_\alpha)} & p_H^{-1}(U_\alpha) \times_G p_K^{-1}(U_\alpha) \\ \downarrow / \Delta_Z^* & & \downarrow / \Delta_Z^* \\ U_\alpha \times Z & \xrightarrow{\mu_\alpha} & (p_H * p_K)^{-1}(U_\alpha) \end{array}$$

(noting that as before we can identify $\frac{Z \times Z}{\Delta_Z^*}$ with Z using addition.)

Now note that for any $\langle \alpha, \beta \rangle \in I^2$ such that $U_\alpha \cap U_\beta \neq \emptyset$ we have the following commutative diagram:

$$\begin{array}{ccccc}
(U_\alpha \cap U_\beta) \times Z \times Z & \xrightarrow{(\varphi_\alpha, \psi_\alpha)} & p_H^{-1}(U_\alpha \cap U_\beta) \times p_K^{-1}(U_\alpha \cap U_\beta) & \xrightarrow{(\varphi_\beta^{-1}, \psi_\beta^{-1})} & (U_\alpha \cap U_\beta) \times Z \times Z \\
\downarrow / \Delta_Z^* & & \downarrow / \Delta_Z^* & & \downarrow / \Delta_Z^* \\
(U_\alpha \cap U_\beta) \times Z & \xrightarrow{\mu_\alpha} & (p_H * p_K)^{-1}(U_\alpha \cap U_\beta) & \xrightarrow{\mu_\beta^{-1}} & (U_\alpha \cap U_\beta) \times Z
\end{array}$$

Going along the top row then down sends $(u, 0, 0)$ to $(u, z_{\alpha, \beta} + g_{\alpha, \beta})$, but going down then along the bottom sends it to $(u, l_{\alpha, \beta})$. Thus $l_{\alpha, \beta} = z_{\alpha, \beta} + g_{\alpha, \beta}$, so our map A_U is a homomorphism. \square

This completes the proof that we have an injective homomorphism $A: Z\text{-Cov}_{\text{def}}(G) \rightarrow \check{H}^1(G, Z)$.

2.5 The map $B: \text{Hom}_{\text{cont}}(\eta(G), Z) \rightarrow \text{Ext}_{\text{def}}(G; Z)$

Given $f \in \text{Hom}_{\text{cont}}(\eta(G), Z)$ we want to find a definable Z -covering homomorphism $B(f) := (p_f: H_f \rightarrow G)$ which is unique up to definable equivalence. Referring to [22] Proposition 1.1.6 we see that our f must factor through one of the co-ordinates in the inverse limit, i.e. though $\text{Ker}(p_H)$ for some $p_H: H \rightarrow G$ in $\text{Cov}_{\text{def}}^0(G)$. This means that, letting l_H denote the projection from $\eta(G)$ to the component $\text{Ker}(p_H)$, there is a homomorphism f_H such that the following diagram commutes:

$$\begin{array}{ccc}
\eta(G) & \xrightarrow{l_H} & \text{Ker } p_H \\
f \downarrow & \swarrow f_H & \\
Z & &
\end{array}$$

Now let

$$H_f^* = \frac{H}{\text{Ker}(f_H)} \times Z,$$

and define p_f^* by letting, for $[x] \in H/\text{Ker}(f_H)$ and $z \in Z$:

$$p_f^*([x], z) := p_H(x).$$

This map, p_f^* , is well defined since, for any $x' \in [x]$ we have that $x' = x + x_0$ for some $x_0 \in \text{Ker}(f_H) \subseteq \text{Ker}(p_H)$. Thus $p_H(x') = p_H(x + x_0) = p_H(x) + p_H(x_0) = p_H(x)$.

Now take

$$H_f = \frac{H_f^*}{\text{Im}(f_H) \times [0]}.$$

and note that p_f^* induces a well defined map

$$p_f : H_f \longrightarrow G,$$

for if $[[x_1], z_1] = [[x_2], z_2]$ then $[x_1] = [x_2]$ so $p_H(x_1) = p_H(x_2)$.

To see that p_f is a definable Z -covering homomorphism notice that p_f is clearly surjective, and the isomorphism in the expression below is canonical, so we can identify $\text{Ker}(p_f)$ and Z .

$$\text{Ker}(p_f) = \frac{(\frac{\text{Ker}(p_H)}{\text{Ker}(f_H)}) \times Z}{\text{Im}(f_H) \times [0]} \cong \frac{\text{Im}(f_H) \times Z}{\text{Im}(f_H) \times [0]} = Z.$$

To show that this map B is well defined we must consider the possibility that the element of the inverse system through which our homomorphism f factored is not unique. So let us suppose that we have the following commutative diagram:

$$\begin{array}{ccc} & \text{Ker}(p_H) & \\ \nearrow l_H & & \searrow f_H \\ \eta(G) & \xrightarrow{f} & Z \\ \searrow l_K & & \nearrow f_K \\ & \text{Ker}(p_K) & \end{array} .$$

We need to show that we have a definable isomorphism of groups between $H_f^* = H/\text{Ker}(f_H) \times Z$ and $K/\text{Ker}(f_K) \times Z = K_f^*$, which commutes with the covering maps p_{H_f} and p_{K_f} .

This reduces to showing we have a definable isomorphism $\tilde{F} : K/\text{Ker}(f_K) \cong H/\text{Ker}(f_H)$ which commutes with the restrictions of the definable covering homomorphisms p_H and p_K to the projections of the first variables, i.e. if $q_H : H/\text{Ker}(f_H) \longrightarrow G$ and $p_K : K/\text{Ker}(f_K) \longrightarrow G$ are given by $[x] \mapsto p_H(x)$ and $[y] \mapsto p_K(y)$ respectively then $q_H = q_K \circ \tilde{F}$.

Note that the diagram also gives, by the first isomorphism theorem,

$$\text{Ker}(p_H)/\text{Ker}(f_H) \cong_{f_H} \text{Im}(f_H) = \text{Im}(f_K) \cong_{f_K^{-1}} \text{Ker}(p_K)/\text{Ker}(f_K) \quad (*).$$

We can always take the fibre product of H and K , and note that $p_{H \times_G K} : H \times_G K \longrightarrow G$ given by $p_{H \times_G K}(x, x') = p_H(x) = p_K(x')$ is in $\text{Cov}_{\text{def}}^0(G)$, and that the following diagram commutes

$$\begin{array}{ccc} & \text{Ker}(p_H) & \\ \nearrow pr & \uparrow l_H & \searrow f_H \\ \text{Ker}(p_{H \times_G K}) & \xleftarrow{l_{H \times_G K}} \eta(G) \xrightarrow{f} & Z \\ \searrow pr' & \downarrow l_K & \nearrow f_K \\ & \text{Ker}(p_K) & \end{array} .$$

Thus, by arguing symmetrically, we can assume that there is a definable covering homomorphism $p: K \rightarrow H$ such that $p_K = p_H \circ p$ and $f_K = f_H \circ p$.

Now let the definable homomorphism $F: K \rightarrow H/\text{Ker}(f_H)$ be the result of applying p and then the projection $H \rightarrow \frac{H}{\text{Ker}(f_H)}$. We prove that $\text{Ker}(F) = \text{Ker}(f_K)$, so that the first isomorphism theorem (for definable groups) gives us the map we need, and we will then see that it commutes with q_H and q_K by the fact that p commutes with p_K and p_H .

We have that $x \in \text{Ker}(F)$ iff $p(x) \in \text{Ker}(f_H)$. But by (*) this then gives $f_K^{-1} \circ f_H(p(x)) \in \text{Ker}(f_K)$, i.e. $x \in \text{Ker}(f_K)$.

Conversely if $x \in \text{Ker}(f_K)$ then $f_K(x) = 0 \in \text{Ker}(p_K)/\text{Ker}(f_K)$, so by (*) again $f_H \circ p(x) = 0 \in \text{Ker}(p_H)/\text{Ker}(f_H)$. So $p(x) \in \text{Ker}(f_H)$, i.e. $x \in \text{Ker}(F)$.

Now it is clear that the definable isomorphism, \tilde{F} , induced by F on the quotient group commutes as required from the following diagram:

$$\begin{array}{ccc}
 K & \xrightarrow{p} & H \\
 \searrow & & \swarrow \\
 \frac{K}{\text{Ker}(f_K)} & \cong \tilde{F} & \frac{H}{\text{Ker}(f_H)} \\
 \downarrow p_K & & \downarrow p_H \\
 & \begin{array}{c} \downarrow q_K \\ \downarrow q_H \end{array} & \\
 & G &
 \end{array}$$

This completes the proof that B is a well defined map.

We do not show that B map is a homomorphism directly, but show that it has an inverse, and that this inverse is a homomorphism.

2.6 The isomorphism $C: \text{Ext}_{\text{def}}(G; Z) \rightarrow \text{Hom}_{\text{cont}}(\eta(G), Z)$

For any definable Z -covering homomorphism $p_H: H \rightarrow G$ we first take the restriction of p_H to the definably connected component of H , $p_{H^0}: H^0 \rightarrow G$. Then, since this is a definable covering homomorphism in $\text{Cov}_{\text{def}}^0(G)$ (by Corollary 3.7 in [6]) there is a canonical projection homomorphism $l_{H^0}: \eta(G) \rightarrow \text{Ker}(p_{H^0})$, which is clearly continuous. Also, since p_H is a definable Z -covering homomorphism, we have $\text{Ker}(p_{H^0}) \leq \text{Ker}(p_H) = Z$, and we call this inclusion g_{H^0} . Thus $g_{H^0} \circ l_{H^0}$ is, in fact, an element of $\text{Hom}_{\text{cont}}(\eta(G), Z)$. Let $C(p_H) = g_{H^0} \circ l_{H^0}$.

To see that C is well-defined we note that if $F: H \rightarrow K$ is a definable equivalence then we have that F restricts to the definably connected components (i.e. $F(e_H) = e_K \in K^0$ so that $F: H^0 \rightarrow K^0$ is a definable equivalence). Then, since $p_H = p_K \circ F$, we get that F maps $\text{Ker}(p_{H^0})$ to

$\text{Ker}(p_{K^0})$, that $F|_{\text{Ker}(p_{H^0})} = \text{id}$, and that F commutes with l_{K^0} and l_{H^0} and also with g_{K^0} and g_{H^0} . Thus the following diagram commutes, so the map is well-defined:

$$\begin{array}{ccccc}
 & & \text{Ker}(p_{H^0}) & & \\
 & \nearrow^{l_{H^0}} & \downarrow F & \searrow^{g_{H^0}} & \\
 \eta(G) & & & & Z \\
 & \searrow_{l_{K^0}} & \downarrow & \nearrow_{g_{K^0}} & \\
 & & \text{Ker}(p_{K^0}) & &
 \end{array}$$

Lemma 2.13 C is a homomorphism.

Proof. We take $p_H: H \rightarrow G$ and $p_K: K \rightarrow G$ in $\text{Ext}_{\text{def}}(G; Z)$ and show that $C(p_H * p_K) = C(p_H) + C(p_K)$. But this follows from the fact that the following diagram commutes.

$$\begin{array}{ccc}
 \eta(G) & \xrightarrow{l_{H^0} \times l_{K^0}} & \text{Ker}(p_{H^0}) \times \text{Ker}(p_{K^0}) \\
 \downarrow l_{H^0} & \searrow^{l_{H^0} \times_G K^0} & \downarrow l_{K^0} \\
 \text{Ker}(p_{H^0}) \times \text{Ker}(p_{K^0}) & \xrightarrow{\cong} & \text{Ker}(p_{H^0} \times_G K^0) \\
 \downarrow g_{H^0} & & \downarrow / \Delta_Z^* \\
 Z \times Z & & \text{Ker}(p_{H^0} *_G K^0) \\
 \downarrow + & \nearrow_{g_{H^0} *_G K^0} & \\
 Z & &
 \end{array}$$

This gives the result since we get $C(p_H * p_K)$ by following the right hand arrows and $C(p_H) + C(p_K)$ by following the left hand arrows.

Lemma 2.14 C is an inverse to B .

Proof. We first show that $C \circ B = \text{id}_{\text{Hom}_{\text{cont}}(\eta(G), Z)}$, so that C is surjective, and then show that C is injective.

So suppose that $f \in \text{Hom}_{\text{cont}}(\eta(G), Z)$ and $B(f) = (p_f: H_f \rightarrow G)$ where $H_f = H/\text{Ker}(f_H) \times Z/\text{Im}(f_H)$, with f factoring through the projection $l_H: \eta(G) \rightarrow \text{Ker}(p_H)$ and $f_H: \text{Ker}(p_H) \rightarrow Z$, for $(p_H: H \rightarrow G) \in \text{Cov}_{\text{def}}^0(G)$. Then note that $H_f^0 = H/\text{Ker}(f_H)$ (as H was definably connected) and that $\text{Ker}(p_{H_f^0}) = \frac{\text{Ker}(p_H)}{\text{Ker}(f_H)}$ so that the following diagram commutes:

$$\begin{array}{ccccc}
 & & & \xrightarrow{l_{H_f^0}} & \\
 & \nearrow & & & \\
 \eta(G) & \xrightarrow{l_H} & \text{Ker}(p_H) & \longrightarrow & \text{Ker}(p_{H_f^0}) \\
 & \searrow f & \downarrow f_H & \nearrow_{g_{H_f^0}} & \\
 & & Z & &
 \end{array}$$

Thus we get that $C(p_f) := g_{H_f^0} \circ l_{H_f^0} = f_H \circ l_H = f$. So C is a left inverse to B .

To see that C is injective let $(p_H : H \rightarrow G) \in \text{Ker}(C)$. Then we have the following diagram

$$\begin{array}{ccc} \eta(G) & \xrightarrow{C(p_H)=0} & Z \\ l_{H^0} \downarrow & \nearrow g_H & \\ \text{Ker}(p_{H^0}) & & \end{array}$$

As $\text{Ker}(p_{H^0}) \neq 0$ implies that $g_H \circ l_{H^0} \neq 0$, we must have $\text{Ker}(p_{H^0}) = 0$, so that there is a definable isomorphism $H^0 \cong_{p_H^0} G$, which gives a definable isomorphism $H \cong G \times Z$, which will clearly commute with the p_H and projection homomorphism $G \times Z \rightarrow G$. But this means that $p_H : H \rightarrow G$ is the identity in the group $\text{Ext}_{\text{def}}(G; Z)$, so C is injective, and thus the inverse of B . \square

Thus we have that C is an isomorphism, and an inverse to B .

Lemma 2.15 *If $f \in \text{Hom}_{\text{cont}}(\eta(G), Z)$ is surjective then $A \circ P \circ B(f) \neq 0$.*

Proof. It is clear from the construction of B that if $f \in \text{Hom}_{\text{cont}}(\eta(G), Z)$ is surjective, and factors through $p_H : H \rightarrow G$, then $\text{Ker}(p_H) = Z$, so that $\text{Im}(f_H) = Z$, and so $H_f = H/\text{Ker}(f_H)$, which is definably connected since H is. Clearly no definably connected definable Z -covering homomorphism can be definably homeomorphic to the trivial Z -covering homomorphism (which is definably disconnected). This means that the map $P \circ B$ will not send any surjective elements of $\text{Hom}_{\text{cont}}(\eta(G), Z)$ to zero. As A is a monomorphism, this gives us that our map $A \circ P \circ B$ has the same property, as required. \square

Thus we have completed the proof of Theorem 1.1.

3 The isomorphism $H^*(G; \mathbb{Q}) \simeq H^*(G/G^{00}; \mathbb{Q})$

Here we prove Theorem 1.2 from the introduction and state and prove some similar applications of the Hurewicz theorem. So throughout this section let \mathcal{N} be a sufficiently saturated o-minimal expansion of an ordered field $(N, 0, 1, +, \cdot, <)$.

Note first that below we will use the o-minimal Čech cohomology theory (\check{H}^*, d^*) constructed in [9] where we also showed that it satisfies the o-minimal Eilenberg-Steenrod axioms. In o-minimal expansions of fields, there is (by [11]) a unique o-minimal cohomology theory with fixed coefficients satisfying the Eilenberg-Steenrod axioms adapted to the o-minimal setting, up to isomorphism. Thus the o-minimal Čech cohomology (\check{H}^*, d^*) coincides with

the o-minimal singular cohomology (H^*, d^*) and so we will switch from one to the other when convenient.

Proof. [of Theorem 1.2]

We know that G is definably compact, definably connected and abelian, and so by [13] G/G^{00} is a compact connected abelian Lie group of dimension $\dim(G)$, and so it is isomorphic to a product of $\dim(G)$ many unit circles in \mathbb{R}^2 . This last group is definable in the real field, which is o-minimal. Thus we can apply our Hurewicz theorem to both G and G/G^{00} , thus obtaining a commutative diagram of homomorphisms:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{cont}}(\eta(G), \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{B_G} & \mathrm{Ext}_{\mathrm{def}}(G, \mathbb{Z}/p\mathbb{Z}) \\ \downarrow \eta(\rho_G) & & \downarrow \overline{\rho_G} \\ \mathrm{Hom}_{\mathrm{cont}}(\eta(G/G^{00}), \mathbb{Z}/p\mathbb{Z}) & \xleftarrow{C_{G/G^{00}}} & \mathrm{Ext}_{\mathrm{top}}(G/G^{00}, \mathbb{Z}/p\mathbb{Z}) \end{array}$$

where $\eta(\rho_G) := C_{G/G^{00}} \circ \overline{\rho_G} \circ B_G$, with $C_{G/G^{00}}$ and B_G as in Section 2, and $\overline{\rho_G}$ is induced by $\rho_G : G \rightarrow G/G^{00}$ (this is well defined since if $h : H \rightarrow G$ is a definable covering homomorphism with kernel $\mathbb{Z}/p\mathbb{Z}$, then $h : H/H^{00} \rightarrow G/G^{00}$ is also a definable covering homomorphism with kernel $\mathbb{Z}/p\mathbb{Z}$ since, by [1] Corollary 4.7, following [15] in the abelian case, H^{00} is torsion free).

Since $\overline{\rho_G}$ is injective, and as we saw above B_G and $C_{G/G^{00}}$ are injective, $\eta(\rho_G)$ is injective. From Theorem 2.3 we know that $\eta(G) = \widehat{\pi_1(G)}$ (profinite completion) so since $\pi_1(G) \simeq \mathbb{Z}^{\dim G}$ (by [10]) we get

$$\eta(G) = \widehat{\mathbb{Z}}^{\dim G} = \Pi_p(\mathbb{Z}_p)^{\dim G}.$$

Similarly we also have

$$\eta(G/G^{00}) = \widehat{\mathbb{Z}}^{\dim G/G^{00}} = \Pi_p(\mathbb{Z}_p)^{\dim G/G^{00}}$$

(where in both cases above \mathbb{Z}_p is the ring of p -adic integers). So using the fact that $\dim G = \dim G/G^{00}$ we see that $\eta(\rho_G)$ is an isomorphism. So B_G , $\overline{\rho_G}$ and $C_{G/G^{00}}$ are also isomorphisms.

Since $H^*(G; \mathbb{Q}) \simeq \wedge[\omega_1, \dots, \omega_{\dim G}]_{\mathbb{Q}}$, with the ω_i 's of degree one (from [10]), by the universal coefficients theorem from homological algebra we also have

$$H^*(G; \mathbb{Z}/p\mathbb{Z}) \simeq \wedge[\omega_1, \dots, \omega_{\dim G}]_{\mathbb{Z}/p\mathbb{Z}}.$$

Thus the Hurewicz homomorphism

$$\mathrm{Hom}_{\mathrm{cont}}(\eta(G), \mathbb{Z}/p\mathbb{Z}) \longrightarrow \check{H}^1(G; \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism, and similarly we have an isomorphism

$$\mathrm{Hom}_{\mathrm{cont}}(\eta(G/G^{00}), \mathbb{Z}/p\mathbb{Z}) \longrightarrow \check{H}^1(G/G^{00}; \mathbb{Z}/p\mathbb{Z}).$$

Combining these two isomorphisms with $\eta(\rho_G)$, and noting that the finitely generated Hopf algebra structure gives that the level 1 cohomology determines the cohomology at all levels, we get an isomorphism

$$H^*(G; \mathbb{Z}/p\mathbb{Z}) \simeq H^*(G/G^{00}; \mathbb{Z}/p\mathbb{Z}).$$

By construction this is induced by the quotient homomorphism $\rho_G : G \rightarrow G/G^{00}$, as we now explain. We will show that the Hurewicz map $\text{Hom}_{\text{cont}}(\eta(G), \mathbb{Z}/p\mathbb{Z}) \rightarrow \check{H}^1(G; \mathbb{Z}/p\mathbb{Z})$ is intrinsic to the group G (i.e. does not rely on any arbitrary choices that we make). This also shows that the maps B_G and $C_{G/G^{00}}$ are canonical. Then we see that the map $\eta(\rho_G)$ is induced by ρ_G , as it is just the composition of the maps B_G and $C_{G/G^{00}}$ and $\overline{\rho_G}$, which is constructed directly from ρ_G . As the final isomorphism is also just a composition of the (canonical) Hurewicz maps and $\eta(\rho_G)$, this will complete the proof that our final isomorphism is indeed induced by ρ_G .

So, to show that the Hurewicz map is intrinsic take a definable isomorphism of groups $f : G \rightarrow G'$. This is a definable covering homomorphism, and so gives a canonical homomorphism

$$\widehat{f} : \text{Hom}_{\text{cont}}(\eta(G), \mathbb{Z}/p\mathbb{Z}) \rightarrow \text{Hom}_{\text{cont}}(\eta(G'), \mathbb{Z}/p\mathbb{Z}).$$

Then we have a commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{\text{cont}}(\eta(G), \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \check{H}^1(G; \mathbb{Z}/p\mathbb{Z}) \\ \downarrow \widehat{f} & & \downarrow f^* \\ \text{Hom}_{\text{cont}}(\eta(G'), \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \check{H}^1(G'; \mathbb{Z}/p\mathbb{Z}) \end{array}$$

where the horizontal arrows are the Hurewicz maps, showing that they are given by the group structure of G alone.

Finally, by the universal coefficients theorem from homological algebra, we then also have an isomorphism

$$H^*(G; \mathbb{Q}) \simeq H^*(G/G^{00}; \mathbb{Q})$$

induced by the quotient homomorphism $\rho_G : G \rightarrow G/G^{00}$ as required. \square

Assume now that G is definably semi-simple, i.e. it has no non trivial normal abelian definable subgroups of positive dimension. In this case, the center $Z(G)$ of G is finite and by [16] and [17] (the proof of Theorem 5.1 there), there are N -definably simple, N -definable groups G_i defined in the pure field N over the emptyset such that $G/Z(G)$ is definably isomorphic to $G_1 \times \cdots \times G_l$.

We will generalize this result in the following way:

Theorem 3.1 *Let G be a definably semi-simple definable group. Then G is definably isomorphic to an N -definably semi-simple, N -definable group G' defined in the pure field N over the emptyset.*

This theorem is a consequence of a more general result which we now introduce. Let \mathcal{N}_0 be a reduct of \mathcal{N} which is still an (o-minimal) expansion of the field N and suppose that \mathcal{S} is an elementary substructure of \mathcal{N}_0 . Following [13] we say that a definable group G has a *very good reduction over \mathcal{S}* if G is definably isomorphic (in \mathcal{N}) to an \mathcal{S} -definable group G' defined with parameters over S .

Proposition 3.2 *If $h : G \rightarrow H$ is a definable covering homomorphism and H has a very good reduction over \mathcal{S} , then there is an \mathcal{S} -definable covering homomorphism $h' : G' \rightarrow H'$ defined with parameters over S which is definably isomorphic (in \mathcal{N}) to $h : G \rightarrow H$. So in particular, G has a very good reduction over \mathcal{S} .*

Proof. By composing h with the definable isomorphism between H and H' (the good reduction of H over \mathcal{S}) we may assume that $H = H'$.

Let $L = h_*(\pi_1(G)) \leq \pi_1(H)$. By the invariance of o-minimal fundamental groups ([2]) we have that $\pi_1(H)$ is the same as $\pi_1(H'(S))$ computed in \mathcal{S} . So L is a subgroup of $\pi_1(H'(S))$. Therefore, working in the o-minimal structure \mathcal{S} (with parameters from S), we see that by [7] Theorem 1.4 as we already saw, there is an \mathcal{S} -definable covering homomorphism $h' : G' \rightarrow H'$ defined over S such that $h'_*(\pi_1(G')) = L$.

Now working in \mathcal{N} it follows from [7] Theorem 1.4, as we already saw, that the definable covering homomorphisms $h : G \rightarrow H$ and $h' : G' \rightarrow H'$ are both definably equivalent to the definable covering homomorphism $\tilde{G}/L \rightarrow H$ induced by the locally definable universal covering homomorphism $\tilde{p} : \tilde{G} \rightarrow G$ of G . In particular, there are definable isomorphisms $G \simeq \tilde{G}/L \simeq G'$. \square

To prove Theorem 3.1 apply Proposition 3.2 to the definable covering homomorphism $h : G \rightarrow G_1 \times \cdots \times G_l$ which is the composition of the quotient definable homomorphism $G \rightarrow G/Z(G)$ and the definable isomorphism $G/Z(G) \simeq G_1 \times \cdots \times G_l$, and take \mathcal{N}_0 to be the pure field N and \mathcal{S} the real algebraic closure of \mathbb{Q} in N .

We can now deduce our theorem for the definably semi-simple case:

Theorem 3.3 *Let G be a definably semi-simple definable group. Then there exists an isomorphism*

$$H^*(G; \mathbb{Q}) \simeq H^*(G/G^{00}; \mathbb{Q})$$

induced by the quotient homomorphism $\rho_G : G \rightarrow G/G^{00}$.

Proof. Let G' be as in Theorem 3.1. Then by [19] Proposition 3.6 and [15] Fact 4.1, we have $G'/(G')^{00} \simeq G'(\mathbb{R})$. So, under the definable isomorphism $G \simeq G'$ of Theorem 3.1, the quotient homomorphism $\rho_G : G \rightarrow G/G^{00}$ can be identified with the standard part homomorphism $\text{st} : G' \rightarrow G'(\mathbb{R})$. Hence, by the invariance of o-minimal singular cohomology from [11], we have an isomorphism

$$H^*(G; \mathbb{Q}) \simeq H^*(G/G^{00}; \mathbb{Q})$$

induced by the quotient homomorphism $\rho_G : G \rightarrow G/G^{00}$ as required. \square

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