O-minimal orientation theory for definable manifolds

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August 3, 2004

Abstract

We develop here an orientation theory for definable manifolds based on the o-minimal site and the o-minimal singular homology.

*With partial support from the FCT (Fundação para a Ciência e Tecnologia), program POCTI (Portugal/FEDER-EU).

1 Introduction

We work over an o-minimal expansion $\mathcal{N} = (N, 0, 1, <, +, \cdot, \dots)$ of a real closed field. Definable means $\mathcal{N}$-definable (possibly with parameters).

We are interested here in orientation theory for definable manifolds based on o-minimal homology theory in analogy to orientation theory of topological manifolds based on topological (singular) homology. For the later, see for example, [d] Chapter VIII. The former already appears in [bo2], however, as we explain below the approach we take here is sightly different. The paper [pt] contains an orientation theory for definable manifolds which does not use o-minimal homology. On definably compact definable manifolds, the three orientation theories agree and they also agree with the topological orientation theory when $N = \mathbb{R}$.

A *definable manifold*, say of dimension n, is a triple $(X, X_i, \phi_i)_{i\in I}$ where $\{X_i : i \in I\}$ is a finite cover of the set X and for each $i \in I$:

(i) we have injective maps $\phi_i : X_i \longrightarrow N^n$ such that $\phi_i(X_i)$ is an open definably connected definable set;

(ii) each $\phi_i(X_i \cap X_j)$ is an open definable subset of $\phi_i(X_i)$;

(iii) the map $\phi_{ij} : \phi_i(X_i \cap X_j) \longrightarrow \phi_j(X_i \cap X_j)$ given by $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ is a definable homeomorphism for all $j \in I$ such that $X_i \cap X_j \neq \emptyset$.

A definable manifold $(X, X_i, \phi_i)_{i\in I}$ is called an *affine definable manifold* if X is a definable subset of some N^k and the topology of X is the induced topology on N^k generated by the open boxes associated to the total ordering on N.

As usual, we will often ignore the *definable charts* $(X_i, \phi_i)_{i\in I}$ of the definable manifold $(X, X_i, \phi_i)_{i\in I}$ and simply say that X is a definable manifold. Based on [bo1], we observe in [e] that a Hausdorff definable manifold is affine. Thus, without loss of generality we will work only with affine definable manifolds.

In the classical case we deal with locally compact topological spaces while here we have to work with definable sets equipped with the o-minimal site instead of the strong topology induced on them by the order topology. This is so because definable sets with the strong topology are totally disconnected and never locally compact unless the definable set is finite or $N = \mathbb{R}$.

The *o-minimal site* DTOP_X on a definable set $X \subseteq N^m$ is the site whose underlying category is the set of all open definable subsets of X (open in the strong topology) with morphisms the inclusions and, an admissible covering $\{U_i : i \in I\}$ of some U in DTOP_X is a family $\{U_i : i \in I\}$ in DTOP_X with $U = \cup\{U_i : i \in I\}$ and such that there are $i_1, \ldots, i_l \in I$ with $U = U_{i_1} \cup \cdots \cup U_{i_l}$.

The other ingredient of our theory is o-minimal homology. In o-minimal expansions of real closed fields, Woerheide gives a direct construction of the o-minimal simplicial and singular homology (H_*, d_*) with coefficients in $\mathbb{Z}$ in [Wo]. This construction easily gives, as in the classical case treated in [d] Chapter VI, Section 7, the o-minimal simplicial and singular homology with coefficients in arbiratry rings.

Woerheide's results are based on the definable triangulation theorem ([vdd]) and on the method of acyclic models from homological algebra and are rather complicated due to the fact that, in arbitrary o-minimal expansions of fields, the classical simplicial approximation theorem and the method of repeated barycentric subdivisions and the Lebesgue number property for the standard simpleces Δ^n fail.

We now explain the novelty of this paper in relation to the theory presented in [bo2]. In [bo2] o-minimal orientation of a definable manifold X of dimension n is defined adapting the classical definition such that although o-minimal homology is used, X is equipped with the strong topology. Our appproch is different mainly because on X we make use of the o-minimal site rather than the strong topology. In this way, o-minimal orientation on X will be defined by the o-minimal orientation sheaf on X.

2 Good covers of definable manifolds

We will from now on assume that $(X, X_i, \phi_i)_{i\in I}$ is a Hausdorff definable manifold of dimension n, hence affine.

By $B_n(p, \epsilon)$ we denote the open ball in N^n centered at p and with radius ϵ; $\overline{B}_n(p, \epsilon)$ will denote the closure of $B_n(p, \epsilon)$ and we set $S^{n-1}(p, \epsilon) = \overline{B}_n(p, \epsilon) - B_n(p, \epsilon)$.

Below we recall the theory from [e] of good covers of definable manifolds by open definable subsets.

Definition 2.1 A *special open definable subset* of X is a definable open subset V of X such that there is a definable homeomorphism $h : V \longrightarrow B_n(0, \epsilon)$.

A *special pair of open definable subsets* of X is a pair (V, U) such that:

(i) V and U are definable open subsets of X with $U \subseteq V$;

(ii) V is a special open definable subset of X;

(iii) there $0 < \delta < \epsilon$ such that $U = h^{-1}(B_n(0, \delta))$ where $h : V \longrightarrow B_n(0, \epsilon)$ is the corresponding definable homeomorphism.

Definition 2.2 Let A be a definable subset of X. A *good cover* of A by special open definable subsets is a family $\{W_1, \dots, W_l\}$ of special open definable subsets of X such that:

(i) the intersection of any two elements of $\{W_1, \dots, W_l\}$ is a finite (possibly empty) union of elements of $\{W_1, \dots, W_l\}$;

(ii) $A \subseteq \cup\{W_i : i = 1, \dots, l\}$.

We say that a good cover $\{W_1, \dots, W_l\}$ of an open definable subset U of X is *compatible with* U if $W_i \subseteq U$ for all $i = 1, \dots, l$.

The following theorem is the main result of [e]. We could not find an analogue of this for topological (or differentiable) manifolds, even in the compact case.

Theorem 2.3 *For every open definable subset U of X, there exists a finite good cover compatible with U. In fact, any finite cover of X by open definable subsets can be refined by a good cover of X by special open definable subsets.*

Let $\{V_i : i = 1, \dots, k\}$ be a good cover of X by special open definable subsets such that $h_i : V_i \longrightarrow B_n(0, \epsilon_i)$, for $i = 1, \dots, k$, is the corresponding definable homeomorphism. If for each $i = 1, \dots, k$, we have $U_i = h_i^{-1}(B_n(0, \delta_i))$ for some $0 < \delta_i < \epsilon_i$, then we say that $\{(V_i, U_i) : i = 1, \dots, k\}$ is a family of special pairs of open definable subsets of X obtained from $\{V_i : i = 1, \dots, k\}$.

Corollary 2.4 *Let $A \subseteq X$ be a definably compact definable subset of X. Then there are finite families $\{A_1, \dots A_l\}$, $\{U_1, \dots, U_k\}$ and $\{V_1, \dots, V_k\}$ such that:*

(1) *$\{A_i : i = 1, \dots, l\}$ is a family of definably compact definable subsets of A with $A = \cup\{A_i : i = 1, \dots, l\}$, and $\{\emptyset, A_1, \dots A_l\}$ is closed under intersection.*

(2) $\{V_i : i = 1, \ldots, k\}$ *is a good cover of* X *by special open definable subsets such that* $\{(V_i, U_i) : i = 1, \ldots, k\}$ *is a family of special pairs of open definable subsets of* X *obtained from* $\{V_i : i = 1, \ldots, k\}$.

(3) *For each* $A_i \neq \emptyset$, *there is* $j_i \in \{1, \ldots, k\}$ *such that* $A_i \subseteq U_{j_i} \subseteq V_{j_i}$ *and for each* $j = 1, \ldots, k$, *there is a definable triangulation* (Ψ_j, M_j) *of* V_j *such that for all* $A_i \subseteq U_j$, $\Psi_j(A_i)$ *is closure of the geometric realization of a simplex of* M_j.

(4) $\{U_i : i = 1, \ldots, k\}$ *is a cover of* A *by open definable subsets such that, for each* $i = 1, \ldots, k$, *the closure* $\overline{U_i}$ *of* U_i *in* X *is definably compact and there is a definable homeomorphism from* $\overline{U_i}$ *into the closed unit ball in* N^n *sending* $\overline{U_i} - U_i$ *into the unit* $(n-1)$*-sphere.*

This corollary is also from [e]. The statment of Corollary 2.4 (4) was obtained in [bo2] by completely different methods.

3 Orientable definable manifolds

In this paper, R will be an hereditary ring with unit, i.e., every submodule of a free R-module is free. All fields are, of course, hereditary. A commutative ring is hereditary if and only if it is a principal ideal domain.

We define here the notion of R-orientation of X. Our approach follows the classical one from [d] Chapter VIII, Section 2. However, unlike in the topological setting, we must work with the o-minimal site and the results from Section 2 will be crucial.

3.1 The local homology R-modules

In the classical case, the results of this subsection are local results. Here they are results about special pairs of open definable subsets. We require these lemmas for our definitions on Subsection 3.2.

Remark 3.1 Let (V, U) be a special pair of open definable subsets of X such that $x \in U$. Then the following hold:

(i) V and U are *acyclic*, i.e., their o-minimal homology is trivial in all degrees except in degree zero (where it is isomorphic to R).

(ii) $V - U$, $V - x$ and $U - x$ have the same o-minimal homology as the $(n-1)$-sphere, i.e., their o-minimal homology is trivial in all degrees except in degrees 0 and $n-1$ where it is isomorphic to R.

Lemma 3.2 *For any point $x \in X$, $H_q(X, X - x; R)$ is R iff $q = n$ and is zero otherwise.*

Proof. By Corollary 2.4, let (V, U) be a special pair of open definable subsets of X such that $x \in U$. Then

$$H_q(X, X - x; R) \simeq H_q(V, V - x; R) \simeq \widetilde{H}_{q-1}(V - x; R) \simeq R.$$

The first equality is obtained by the o-minimal excision axiom for the closed definable subset $X - V$ of the definable open set $X - x$ and the second follows from the exact sequence of the pair $(V, V - x)$ since V is acyclic (see Remark 3.1). The third equality follows from Remark 3.1. $\square$

Lemma 3.2 shows that a definable manifold X of dimension n is a *homology definable manifold* of dimension n, i.e., X is a definable set of dimension n such that for every $x \in X$, the ring $H_q(X, X - x; R)$ is R if and only if $q = n$ and is zero otherwise. For a different proof of this fact see [bo2] Lemma 4.10.

Definition 3.3 Suppose that $\alpha_x \in H_n(X, X - x; R)$ and $U \subseteq X$ is an open definable neighbourhood of x. Then $\alpha \in H_n(X, X - U; R)$ is called a *continuation of α_x in U* if we have $\alpha_x = j_x^U(\alpha)$ where

$$j_x^U : H_n(X, X - U; R) \longrightarrow H_n(X, X - x; R)$$

is the canonical homomorphism induced by inclusion.

Arguing as in [d] Chapter VIII, Lemma 2.2, one can show that for $\alpha_x \in H_n(X, X - x; R)$ there exists an open definable neighbourhood U of x and a continuation $\alpha \in H_n(X, X - U; R)$ of α_x in U. However we will be interested in a special kind of continuations.

Lemma 3.4 *Let (V, U) be a special pair of open definable subsets of X such that $x \in U$. Then for every $y \in U$, the homomorphism*

$$j_y^U : H_n(X, X - U; R) \longrightarrow H_n(X, X - y; R)$$

is an isomorphism (hence α_y has a unique continuation in U). In particular, if α_x is a generator of $H_n(X, X-x; R)$ and α is the unique continuation of α_x in U, then for every $y \in U$, $j_y^U(\alpha)$ is a generator of $H_n(X, X-y; R)$.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccc} H_n(X, X-U; R) & \simeq & H_n(V, V-U; R) & \simeq & \widetilde{H}_{n-1}(V-U; R) \\ \downarrow^{j_y^U} & & \downarrow & & \downarrow \\ H_n(X, X-y; R) & \simeq & H_n(V, V-y; R) & \simeq & \widetilde{H}_{n-1}(V-y; R) \end{array}$$

in which the left horizontal isomorphism are excisions and the right ones are connecting homomorphisms (V is acyclic). The right vertical arrow is an isomorphism by Remark 3.1. Therefore, j_y^U is an isomorphism. □

3.2 The orientation sheaf of a definable manifold

As we mentioned before, here we make use of the o-minimal site. Our definition are obtained from the classical ones by replacing the use of the topology by the o-minimal site. We will use the words presheafs, étalé spaces and sheafs, but our definitions are self contained and no knowledge of the theory of sheafs is required.

Definition 3.5 The *R-orientation presheaf O^X on X* is the contravariant functor from the category DTOP_X into the category of R-modules such that U in DTOP_X is mapped into $H_n(X, X-U; R)$ and the inclusion $V \longrightarrow U$ in DTOP_X is mapped into the homomorphism of R-modules

$$j_V^U : H_n(X, X-U; R) \longrightarrow H_n(X, X-V; R)$$

induced by inclusion. The homomorphism j_V^U is called a restriction map.

Note that, if $V \longrightarrow U$ is in DTOP_X and $y \in V$, then $j_y^V \circ j_V^U = j_y^U$. Therefore, by Lemma 3.4 the *stalk O_x^X of O^X at $x \in X$*, which is by definition the direct limit $O_x^X = \lim_{x \in U} O^X(U)$ with respect to the restriction maps, is $H_n(X, X-x; R)$ and, $j_x^U : H_n(X, X-U; R) \longrightarrow H_n(X, X-x; R)$ is exactly the natural map induced by the direct limit.

Definition 3.6 The *étalé space* $\widetilde{O}^X$ of the R-orientation presheaf O^X is the topological space

$$\widetilde{O}^X = \{(x, \alpha_x) : x \in X,\ \alpha_x \in H_n(X, X - x; R)\}$$

where the basis for the topology on $\widetilde{O}^X$ is given by $(W, \alpha_W) = \{(x, \alpha_x) : x \in W,\ \alpha_x = j_x^W(\alpha_W)\}$ for open definable subsets W of X for which there is a special pair (V, W) of open definable subsets of X together with the *étalé map* $p^X : \widetilde{O}^X \longrightarrow X$ given by $p^X(x, \alpha_x) = x$ (i.e., p^X is locally a homeomorphism).

The fact that the collection of open subsets of $\widetilde{O}^X$ given above is a basis for a topology follows from Lemma 3.4. Also, by the same lemma, if $(x, \alpha_x) \in \widetilde{O}^X$, (V, U) is a special pair of open definable subsets of X with $x \in U$ and α_U is the unique continuation of α_x to U, then $(p^X)_| : (U, \alpha_U) \longrightarrow U$ is a homeomorphism .

Definition 3.7 Let U be a nonempty definable subset of X. A map $s : U \longrightarrow \widetilde{O}^X$ such that $p^X \circ s = 1_U$ is called a *section (of $\widetilde{O}^X$) over U* if for every special pair (V, W) of open definable subsets of X such that $U \cap W \neq \emptyset$, for each definably connected component Y of $U \cap W$, there is a basis open subset (W, α_W^Y) of $\widetilde{O}^X$ such that $s(y) = (y, j_y^W(\alpha_W^Y))$ for all $y \in Y$.

Example 3.8 Assume that $h : X \longrightarrow B_n(p, \epsilon)$ is a definable homeomorphism into the open ball in N^n.

Let α be an element of $H_n(X, X - h^{-1}(p); R)$. For each $0 < r < \epsilon$, let $\alpha_r \in H_n(X, X - U_r; R)$ be the unique continuation of α to U_r, where $U_r = h^{-1}(B_n(p, r))$. Let $s_\alpha : X \longrightarrow \widetilde{O}^X$ be given by $s_\alpha(x) = (x, j_x^{U_r}(\alpha_r))$ whenever $x \in U_r$.

Clearly, s_α is a well defined map such that $p \circ s_\alpha = 1_X$, since if $r < t < \epsilon$, then $\alpha_r = j_{U_r}^{U_t}(\alpha_t)$. Also, s_α is in $\Gamma(X; R)$. In fact, if (V, W) is a special pair of open definable subsets of X, then there is $0 < r < \epsilon$ with $V \subseteq U_r$ and $\alpha_W = j_W^{U_r}(\alpha_r)$ is such that $s_\alpha(y) = (y, j_y^W(\alpha_W))$ for all $y \in W$.

For the classical analogue of Definition 3.7 see [d] Chapter VIII, Definition 2.4. There, a section is a continuous map $s : U \longrightarrow \widetilde{O}^X$ such that $p^X \circ s = 1_U$. Later in this subsection, we will show that, if $N = \mathbb{R}$ then a map $s : U \longrightarrow \widetilde{O}^X$ such that $p^X \circ s = 1_U$ is a section if and only if is continuous.

Proposition 3.9 *Let U be a nonempty definable subset of X. Then the set $\Gamma(U;R)$ of all sections over U is in a natural way an R-module.*

Proof. For $s,t \in \Gamma(U;R)$, let $s+t : U \longrightarrow \widetilde{O}^X$ be the map given by $(s+t)(u) = (u, s'(u)+t'(u))$ for all $u \in U$, where s', t' are such that $s(u) = (u, s'(u))$ and $t(u) = (u, t'(u))$ for all $u \in U$. Clearly we have $p^X \circ (s+t) = 1_U$.

Let (V, W) be a special pair of open definable subsets of X such that $U \cap W \neq \emptyset$ and let Y be a definably connected component of $U \cap W$. Let $u \in Y$ and suppose that $(s+t)(u) \in (W, \alpha_W)$. By Lemma 3.4, there are unique α_W^s, α_W^t such that $s'(u) = j_u^W(\alpha_W^s)$ and $t'(u) = j_u^W(\alpha_W^t)$. Since $j_u^W(\alpha_W) = j_u^W(\alpha_W^s) + j_u^W(\alpha_W^t)$, by Lemma 3.4, $\alpha_W = \alpha_W^s + \alpha_W^t$. This implies that $(s+t)(w) = (w, j_w^W(\alpha_W))$ for all $w \in Y$.

For $s \in \Gamma(U;R)$ and $r \in R$, let $rs : U \longrightarrow \widetilde{O}^X$ be the map given by $rs(u) = (u, rs'(u))$ for all $u \in U$, where s' is such that $s(u) = (u, s'(u))$ for all $u \in U$. Clearly we have $p^X \circ (rs) = 1_U$. Let (V, W) be a special pair of open definable subsets of X such that $U \cap W \neq \emptyset$ and let Y be a definably connected component of $U \cap W$. Let $u \in Y$ and suppose that $rs(u) \in (W, \alpha_W)$. By Lemma 3.4 there is a unique α_W^s such that $s'(u) = j_u^W(\alpha_W^s)$. Since $j_u^W(\alpha_W) = rj_u^W(\alpha_W^s)$, by Lemma 3.4, $\alpha_W = r\alpha_W^s$. This implies that $rs(w) = (w, j_w^W(\alpha_W))$ for all $w \in Y$. □

Before we proceed we point out the following result which will be very useful.

Lemma 3.10 *Suppose that U is a nonempty definable subset of X and let $s, r \in \Gamma(U;R)$. Then the subset of U on which s and r coincide is a finite (possible empty) union of definably connected components of U.*

Proof. Let u and v be two points in a definably connected component of U. Then there is a definable path in U from u to v. The image B of this definable path is definably compact and $s_{|B}, r_{|B} \in \Gamma(B;R)$. By Corollary 2.4, let $(V_1, U_1), \ldots, (V_k, U_k)$ be special pairs of open definable subsets of X such that $B \subseteq \cup\{U_1 : i = 1, \ldots, k\}$. By definition, for each $= 1, \ldots, k$, $s_{|B}$ (resp., $r_{|B}$) extends uniquely to each definably connected component of $U_i \cap B$. Since B is definably connected, these components cover B. Thus s and r coincide in a point in a definably connected component of U if and only if they coincide on that definably connected component. □

Remark 3.11 Assume that $h : X \longrightarrow B_n(p, \epsilon)$ is a definable homeomorphism into the open ball in N^n. Then it follows from Lemma 3.10 that all sections of $\Gamma(X; R)$ are of the form s_α as in Example 3.8. Hence, we have an obvious isomorphism $\Gamma(X; R) \simeq R$ of R-modules.

For the topological analogue of the next theorem see [d] Chapter VIII, (2.7).

Proposition 3.12 *Let U be a nonempty definable subset of X. Then we have a canonical homomorphism*

$$j_U : H_n(X, X - U; R) \longrightarrow \Gamma(U; R)$$

defined by $j_U(\alpha)(x) = (x, j^U_x(\alpha))$ for $x \in U$. Moreover, if $V \subseteq U$ is also a nonempty definable subset of X, then we have the commutative diagram

$$\begin{array}{ccc} H_n(X, X - U; R) & \stackrel{j_U}{\rightarrow} & \Gamma(U; R) \\ \downarrow j^U_V & & \downarrow r^U_V \\ H_n(X, X - V; R) & \stackrel{j_V}{\rightarrow} & \Gamma(V; R) \end{array}$$

where r^U_V is the restriction map.

Proof. Let $\alpha \in H_n(X, X - U; R)$. It is clear that $p^X \circ j_U(\alpha) = 1_U$. Let (V, W) be a special pair of open definable subsets of X such that $W \cap U \neq \emptyset$ and let Y be a definably connected component of $W \cap U$.

Let $w \in Y$ and, by Lemma 3.4, let $\alpha^Y_W \in H_n(X, X - W; R)$ be the unique element such that $j^W_w(\alpha^Y_W) = j^U_w(\alpha)$. Let $y \in Y$. Then there is a definable path in Y from w to y. The image of this definable path can be decomposed into cells. Clearly, we can assume without loss of generality that w and y are the endpoint of the closure C one such cell.

Since W is definably homeomorphic to an open ball in N^n, we can identify W with an open definable subset of the $(n+1)$-sphere $S = S^{n+1}$ in N^{n+1}. Furthermore, under this identification, C in S is definably homeomorphic to a closed interval. We have, by the excision axiom for o-minimal homology, $H_q(S - c, S - C; R) = H_q(W - c, W - C; R)$ for all $q \in \mathbb{Z}$ and $c \in C$. By [Wo], $S - C$ and $S - c$ have the o-minimal homology of a point. Thus, by the exactness axiom, $H_q(W - c, W - C; R) = 0$ for all $q \in \mathbb{Z}$. It follows that the inclusion $W - C$ into $W - c$ induces an isomorphism in homology and hence the inclusion $(W, W-C) \longrightarrow (W, W-c)$ induces an isomorphism $H_q(W, W-C; R) \longrightarrow H_q(W, W-c; R)$ for all $q \in \mathbb{Z}$. By excision, we have an isomorphism $j_c^C : H_n(X, X-C; R) \longrightarrow H_n(X, X-c; R)$ for all $c \in C$. Since j_w^C is an isomorphism, we have $j_C^W(\alpha_W^Y) = j_C^U(\alpha)$. Hence, for all $c \in C$, we have $j_c^W(\alpha_W^Y) = j_c^C(j_C^W(\alpha_W^Y)) = j_c^C(j_C^U(\alpha)) = j_c^U(\alpha)$ as required. □

The argument in the second paragraph of the proof of Proposition 3.12 is basically the same as that of Case 1 in the proof of [bo2] Theorem 5.2.

Remark 3.13 Suppose that $V \subseteq U \subseteq X$ are nonempty definable subsets of X. Let $\Gamma(U, V; R)$ be the kernel of the restriction homomorphism $r_V^U : \Gamma(U; R) \longrightarrow \Gamma(V; R)$. Then there is a unique homomorphism $j_{U,V}$ making the following diagram commutative

$$\begin{array}{ccccccc} & & H_n(X-V, X-U; R) & \rightarrow & H_n(X, X-U; R) & \stackrel{j_V^U}{\rightarrow} & H_n(X, X-V; R) \\ & & \downarrow j_{U,V} & & \downarrow j_U & & \downarrow j_V \\ 0 & \rightarrow & \Gamma(U, V; R) & \rightarrow & \Gamma(U; R) & \stackrel{r_V^U}{\rightarrow} & \Gamma(V; R). \end{array}$$

Corollary 3.14 *Suppose that U is a nonempty definable subset of X and $N = \mathbb{R}$. Then a map $s : U \longrightarrow \widetilde{O}^X$ such that $p^X \circ s = 1_U$ is a section if and only if is continuous.*

Proof. Let $s \in \Gamma(U; R)$ and let $u \in U$ be such that $s(u) \in (W, \alpha_W)$. Then there is a special pair (V, W) of open definable subsets of X with $U \cap W \neq \emptyset$. Let Y be the definably connected component of $U \cap W$ such that $u \in Y$. Then Y is an open definable neighbourhood of u and, by definition, $s(Y) \subseteq (W, \alpha_W)$. So $s : U \longrightarrow \widetilde{O}^X$ is a continuous map.

Let $s : U \longrightarrow \widetilde{O}^X$ be a continuous map such that $p^X \circ s = 1_U$. Let (V, W) be a special pair of open definable subsets of X with $U \cap W \neq \emptyset$. Let Y be a definably connected component of $U \cap W$ and $u \in Y$. By Lemma 3.4 there is a unique $\alpha \in H_n(X, X-W; R)$ such that $s(u) = (u, j_u^W(\alpha))$. By Proposition 3.12, $j_W(\alpha)_{|Y} \in \Gamma(Y; R)$ and, by the first paragraph, $j_W(\alpha)_{|Y} : Y \longrightarrow \widetilde{O}^X$ is a continuous map such that $p^X \circ j_W(\alpha)_{|Y} = 1_Y$. Since Y is connected and $s(u) = j_W(\alpha)_{|Y}(u)$, by [d] Chapter VIII, (2.6), we have $s_{|Y} = j_W(\alpha)_{|Y}$. Thus, $s \in \Gamma(U; R)$ as required. □

We now construct the R-orientation sheaf associated to the R-orientation presheaf.

Definition 3.15 The étalé space $\widetilde{O}^X$ determines the presheaf of sections which maps U in DTOP_X into $\Gamma(U;R)$ and an inclusion $V \longrightarrow U$ in DTOP_X into the restriction map $r^U_V : \Gamma(U;R) \longrightarrow \Gamma(V;R)$. Futhermore, we have a morphism of presheafs $j : O^X \longrightarrow \widetilde{O}^X$ given by $j_U : O^X(U) \longrightarrow \widetilde{O}^X(U)$.

Proposition 3.16 *The presheaf $\widetilde{O}^X$ is a sheaf, called the R-orientation sheaf of X, i.e., for every cover $\{U_i : i \in I\}$ in DTOP_X of U in DTOP_X the following conditions hold:*

(i) if $s, t \in \widetilde{O}^X(U)$ and $r^U_{U_i}(s) = r^U_{U_i}(t)$ for all $i \in I$, then $s = t$;

(ii) if $s_i \in \widetilde{O}^X(U_i)$ are such that $r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j)$ for every $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, then there exists $s \in \widetilde{O}^X(U)$ such that $r^U_{U_i}(s) = s_i$ for all $i \in I$.

Proof. Since sections are maps, (i) is clear. For (ii), let $s : U \longrightarrow \widetilde{O}^X$ be the map given by $s(u) = s_i(u)$ for $u \in U_i$. Clearly, s is well defined and $p^X \circ s = 1_U$. It remains to show that $s \in \Gamma(U;R)$. For this we may clearly assume that $U = \cup\{U_i : i \in I\}$.

Let (V, W) be a special pair of open definable subsets of X such that $U \cap W \neq \emptyset$ and let Y be a definably connected component of $U \cap W$. Let $I(W) = \{i \in I : U_i \cap W \neq \emptyset\}$. Then there are definably connected open definable subsets $Y_1, \dots, Y_l$ of X such that $Y = Y_1 \cup \dots \cup Y_l$ and each Y_j is a definably connected component of $U_{i(j)} \cap W$ for some $i(j) \in I(W)$. For each $j = 1, \dots, l$, by definition, there exists a basis open subset $(W, \alpha^{Y_j}_W)$ of $\widetilde{O}^X$ such that $s(y) = s_{i(j)}(y) = (y, j^W_y(\alpha^{Y_j}_W))$ for all $y \in Y_j$. Let $Z = \cup\{Y_j : \alpha^{Y_j}_W = \alpha^{Y_1}_W\}$. Then Z and $Y \setminus Z$ are open definable subsets of Y. As Y is definably connected and Z is nonempty, we have $Y = Z$ and therefore, if $\alpha^Y_W = \alpha^{Y_1}_W$, we have $s(y) = (y, j^W_y(\alpha^Y_W))$ for all $y \in Y$ as required. $\square$

Since, by Theorem 2.3, X is covered by finitely many special open definable subsets, we can often combine Example 3.8 and Proposition 3.16 to construct element of $\Gamma(X;R)$.

3.3 Orientable definable manifolds

As before, we follow here the classical theory from [d] Chapter VIII, Section 2. But we work in the setup of the previous subsection.

Definition 3.17 If $x \in X$, then an R-orientation of X at x is a generator α_x of the R-module $H_n(X, X - x; R)$ (see Lemma 3.2).

An *R-orientation of* X is a section $s \in \Gamma(X; R)$ such that for each $x \in X$, if $s(x) = (x, s'(x))$, then $s'(x)$ is an R-orientation of X at x.

We say that X is *R-orientable* if there is an R-orientation of X. Finally, X is *orientable* if X is $\mathbb{Z}$-orientable.

Example 3.18 Assume that $h : X \longrightarrow B_n(p, \epsilon)$ is a definable homeomorphism into the open ball in N^n. If α is a generator of $H_n(X, X - h^{-1}(p); R)$, then the section $s_\alpha \in \Gamma(X; R)$ constructed in Example 3.8 is an R-orientation of X.

By the universal coefficient theorem for o-minimal singular homology (compare with [d] Chapter VI (7.9)), if X is orientable then it is R-orientable for all coefficient rings R. Also, the excision axiom implies the following observation.

Remark 3.19 If X is R-orientable and $Y \subseteq X$ is an open definable subset, then Y is R-orientable. In particular, X is R-orientable if and only if all its definably connected components are R-orientable.

The Künneth formula for homology for o-minimal singular homology (compare with [d] Chapter VII (2.6)) easily implies the following remark. For details compare with [d] Chapter VIII, 2.13.

Remark 3.20 Let X and Y be definable manifolds. Then there is a map $\mu : \widetilde{O}^X \times \widetilde{O}^Y \longrightarrow \widetilde{O}^{X \times Y}$ such that and the following diagram

$$\begin{array}{ccc} \widetilde{O}^X \times \widetilde{O}^Y & \stackrel{\mu}{\rightarrow} & \widetilde{O}^{X\times Y} \\ \downarrow^{p^X \times p^Y} & & \downarrow^{p^{X\times Y}} \\ X \times Y & \stackrel{1_{X\times Y}}{\rightarrow} & X \times Y \end{array}$$

is commutative. This map determines an R-module homomorphism

$$\gamma : \Gamma(X; R) \times \Gamma(Y; R) \longrightarrow \Gamma(X \times Y; R)$$

which sends an R-orientation of X and of Y into an R-orientation of $X \times Y$.

The next remark follows immediatly from Lemma 3.10.

Remark 3.21 Suppose that X is definably compact and definably connected. Then any two R-orientations on X that agree at one point are equal. In particular, X has a unique $\mathbb{Z}/2\mathbb{Z}$-orientation and if X is orientable then it has exactly two distinct orientations.

Proposition 3.22 *Suppose that R has no zero divisors. If X is R-orientable and has k definably connected components then $\Gamma(X;R) \simeq R^k$.*

Proof. Clearly, we may assume that X is definably connected. Suppose that $s \in \Gamma(X;R)$ is an R-orientation of X. Fix $x \in X$ and suppose that $s(x) = (x, s'(x)) \in (W, \alpha_W)$. Since $s'(x)$ is an R-orientation at x, by Lemma 3.4, α_W is a generator of $H_n(X, X-W;R) \simeq R$.

Let $t \in \Gamma(X;R)$. Then $t(x)$ is in a unique (W, α_W^t) and, by assumption on R, there is a unique $u(t) \in R$ such that $\alpha_W^t = u(t)\alpha_W$. But then $t(x) = j_W(\alpha_W^t)(x) = u(t)j_W(\alpha_W)(x) = u(t)s(x)$ and, by Lemma 3.10, we have $t = u(t)s$. By uniqueness of $u(t)$, if $r \in R$ and $t, l \in \Gamma(X;R)$, then $u(rt) = ru(t)$ and $u(t+l) = u(t) + u(l)$. Hence, we have a well defined isomorphism $u : \Gamma(X;R) \longrightarrow R$ of R-modules. □

Remark 3.23 Suppose that R has no zero divisors. If X is definably connected and $s \in \Gamma(X;R)$ is an R-orientation of X, then the étalé map $p^X : \widetilde{O}^X \longrightarrow X$ is trivial, i.e., there is a bijection $\phi : \widetilde{O}^X \longrightarrow X \times R$ given by $\phi(x, \alpha_x) = (x, \lambda(x))$ with $\alpha_x = \lambda(x)s'(x)$ where $s(x) = (x, s'(x))$ for all $x \in X$, such that for each $r \in R$, $\phi^{-1}(X \times \{r\})$ is a definable manifold and $p^X_{|} : \phi^{-1}(X \times \{r\}) \longrightarrow X$ is a definable homeomorphism.

We end the section with a useful characterization of the notion of R-orientability. Below, we talk of definable covering maps and definable covering maps. A continuous definable map $q : Y \longrightarrow X$ between definable sets is called a *definable covering map* if q is surjective and there is a finite family $\{U_l : l \in L\}$ of open definable subsets of X such that $X = \cup\{U_l : l \in L\}$ and, for each $l \in L$, the definable subset $q^{-1}(U_l)$ of Y is a disjoint union of open definable subsets of Y, each of which is mapped homeomorphically by q onto U_l.

Proposition 3.24 *If X is definably compact and definably connected, then there is a 2-fold definable covering map $q : E \longrightarrow X$ such that E is an* *orientable definable manifold and E is definably connected if and only if X is non orientable.*

Proof. By Corollary 2.4, let $(V_1, U_1), \ldots, (V_k, U_k)$ be special pairs of open definable subsets of X such that $X \subseteq \cup\{U_i : i = 1, \ldots, k\}$. For each $i = 1, \ldots, k$, let $h_i : V_i \longrightarrow B_n(0, \epsilon_i)$ be the corresponding definable homeomorphism such that $U_i = h_i^{-1}(B_n(0, \delta_i))$. For each $i = 1, \ldots, k$, let $\delta_i < \theta_i < \epsilon_i$ and $W_i = h_i^{-1}(B_n(0, \theta_i))$. Of course, (V_i, W_i) and (W_i, U_i) are special pairs of open definable subset of X for $i = 1, \ldots, k$.

Let E be the set of pairs (x, α_x), where $x \in X$ and α_x is one of the generators of $H_n(X, X - x; \mathbb{Z})$ (see Lemma 3.2). We denote by α_x^- and α_x^+ the two generators of $H_n(X, X - x; \mathbb{Z})$.

For each $i \in I$, $Y_i \in \{U_i, W_i\}$ and $\star \in \{-, +\}$, by Lemma 3.4, let $\alpha_{Y_i}^\star$ be the two generators of $H_n(X, X - Y_i; \mathbb{Z})$ and let $Y_i^\star = (Y_i, \alpha_{Y_i}^\star) \subseteq \widetilde{O}^X$. As we saw on page 8, Lemma 3.4 implies that, for $i \in I$ and $\star \in \{-, +\}$, the restriction $p_|^X : Y_i^\star \longrightarrow Y_i$ of the étalé map $p^X : \widetilde{O}^X \longrightarrow X$ is a homeomorphism. On the other hand, again by Lemma 3.4, $Y_i^\star \subseteq E$. Thus E is a definable manifold with charts $\{(W_i^\star, \psi_i^\star) : i \in I \text{ and } \star \in \{-, +\}\}$ where $\psi_i^\star : W_i^\star \longrightarrow B_n(0, \theta_i)$ is the restriction of $h_{i|W_i} \circ p_|^X$ to $W_i^\star$. Furthermore, the definable map $q : E \longrightarrow X$ given by $q(x, \alpha_x) = x$ is a 2-fold definable covering map.

Note also that $\{(W_i^\star, U_i^\star) : i = 1, \ldots, k \text{ and } \star \in \{-, +\}\}$ are special pairs of open definable subsets of E such that $E \subseteq \cup\{U_i^\star : i = 1, \ldots, k$ and $\star \in \{-, +\}\}$.

For $i = 1, \ldots, k$ and $*, \star \in \{-, +\}$, let $x^* = (x, \alpha_x^*) \in U_i^\star$ and consider the following commutative diagram:

$$\begin{array}{ccc}
H_n(E, E - U_i^\star; \mathbb{Z}) & \overset{H_n(q)}{\rightarrow} & H_n(X, X - U_i; \mathbb{Z}) \\
\downarrow j_{x^*}^E & & \downarrow j_x^X \\
H_n(E, E - x^*; \mathbb{Z}) & \overset{H_n(q)}{\rightarrow} & H_n(X, X - x; \mathbb{Z}) \\
\uparrow & & \uparrow \\
H_n(U_i^\star, U_i^\star - x^*; \mathbb{Z}) & \overset{H_n(q_|)}{\rightarrow} & H_n(U_i, U_i - x; \mathbb{Z})
\end{array}$$

where the vertical arrows are induced by the inclusions. By Lemma 3.4 and excision, the vertical arrows are isomorphisms. Since $q_| : U_i^\star \longrightarrow U_i$ is a definable homeomorphism, the bottom arrow is also an isomorphism. Thus the top arrow is an isomorphism as well.

For $i = 1, \ldots, k$ and $\star \in \{-,+\}$, let $\beta_i^\star = (H_n(q))^{-1}(\alpha_i^\star)$ and $s_i^\star = j_{U_i^\star}(\beta_i^\star) \in \Gamma(U_i^\star;\mathbb{Z})$ (by Proposition 3.12). By the diagram above, we have $r^{U_i^\star}_{U_i^\star \cap U_j^*}(s_i^\star) = r^{U_j^*}_{U_i^\star \cap U_j^*}(s_j^*)$ for every $i, j = 1, \ldots, k$ and $*, \star \in \{-,+\}$ such that $U_i^\star \cap U_j^* \neq \emptyset$. Therefore, by Proposition 3.16, there exists $s \in \Gamma(E;\mathbb{Z})$ such that $r^E_{U_i^\star}(s) = s_i^\star$ for all $i = 1, \ldots, k$ and $\star \in \{-,+\}$. Clearly, s is an orientation of E.

Suppose that X is non orientable. Then E is definably connected for otherwise, q induces a definable homeomorphism between X and a definably connected component of E contradicting the orientability of E and its definably connected components. Conversely, if $s \in \Gamma(X;\mathbb{Z})$ is an $\mathbb{Z}$-orientation of X, then $s : X \longrightarrow E$ is a definable continuous map. By construction of E, both $s(X)$ and $E - s(X)$ are nonempty open definable subsets of E. Therefore, E has two definably connected components since $E - s(X)$ coincides with $-s(X)$. □

Proposition 3.25 *Suppose that R has no zero divisors. If X is definably connected, definably compact and non R-orientable then* $\Gamma(X;R) \simeq \{r \in R : 2r = 0\}$.

Proof. Since X is non R-orientable, by the universal coefficient theorem for o-minimal singular homology (compare with [d] Chapter VI (7.9)), X is non orientable. Thus, by Proposition 3.24, there is a 2-fold definable covering map $q : E \longrightarrow X$ such that E is a definably connected orientable definable manifold. By the universal coefficient theorem for o-minimal singular homology, there is an R-orientation $s \in \Gamma(E;R)$ of E determined by an orientation of E.

As in the proof of Proposition 3.24, for $i = 1, \ldots, k$, $*, \star \in \{-,+\}$ and $x^* = (x, \alpha_x^*) \in U_i^\star$ consider the following commutative diagram of isomorphisms:

$$\begin{array}{ccc}
H_n(E, E - U_i^\star; R) & \stackrel{H_n(q)}{\rightarrow} & H_n(X, X - U_i; R) \\
\downarrow j^E_{x^*} & & \downarrow j^X_x \\
H_n(E, E - x^*; R) & \stackrel{H_n(q)}{\rightarrow} & H_n(X, X - x; R) \\
\uparrow & & \uparrow \\
H_n(U_i^\star, U_i^\star - x^*; R) & \stackrel{H_n(q_|)}{\rightarrow} & H_n(U_i, U_i - x; R)
\end{array}$$

where the vertical arrows are induced by the inclusions.

For $t \in \Gamma(X;R)$, $i = 1, \ldots, k$ and $\star \in \{-,+\}$, let $\beta_i^\star = (H_n(q))^{-1}(\tau_i)$ and $\epsilon(t)_i^\star = j_{U_i^\star}(\beta_i^\star) \in \Gamma(U_i^\star;R)$ (by Proposition 3.12), where $t_{|U_i} = j_{U_i}(\tau_i)$. By the diagram above, we have $r^{U_i^\star}_{U_i^\star \cap U_j^*}(\epsilon(t)_i^\star) = r^{U_j^*}_{U_i^\star \cap U_j^*}(\epsilon(t)_j^*)$ for every $i, j = 1, \ldots, k$ and $*, \star \in \{-,+\}$ such that $U_i^\star \cap U_j^* \neq \emptyset$. Therefore, by Proposition 3.16, there exists $\epsilon(t) \in \Gamma(E;R)$ such that $r^E_{U_i^\star}(\epsilon(t)) = \epsilon(t)_i^\star$ for all $i = 1, \ldots, k$ and $\star \in \{-,+\}$.

The map $\epsilon : \Gamma(X;R) \longrightarrow \Gamma(E;R)$ is clearly an injective homomorphism of R-modules. Our goal is to show that $u \circ \epsilon : \Gamma(X;R) \longrightarrow \{r \in R : 2r = 0\}$ is an isomorphism, where $u : \Gamma(E;R) \longrightarrow R$ is the isomorphism of R-modules given in (the proof of) Proposition 3.22.

Let $m : E \longrightarrow E$ be the definable homeomorphism given by $m(x, \alpha_x^+) = (x, \alpha_x^-)$ and $m(x, \alpha_x^-) = (x, \alpha_x^+)$. Then $q = q \circ m$. For $t \in \Gamma(E;R)$, $i = 1, \ldots, k$ and $\star \in \{-,+\}$, let $\beta_i^\star = (H_n(m))^{-1}(\tau_i^\star)$ and $\mu(t)_i^\star = j_{U_i^\star}(\beta_i^\star) \in \Gamma(U_i^\star;R)$ (by Proposition 3.12), where $t_{|U_i^\star} = j_{U_i^\star}(\tau_i^\star)$. By construction, we have $r^{U_i^\star}_{U_i^\star \cap U_j^*}(\mu(t)_i^\star) = r^{U_j^*}_{U_i^\star \cap U_j^*}(\mu(t)_j^*)$ for every $i, j = 1, \ldots, k$ and $*, \star \in \{-,+\}$ such that $U_i^\star \cap U_j^* \neq \emptyset$. Therefore, by Proposition 3.16, there exists $\mu(t) \in \Gamma(E;R)$ such that $r^E_{U_i^\star}(\mu(t)) = \mu(t)_i^\star$ for all $i = 1, \ldots, k$ and $\star \in \{-,+\}$.

The map $\mu : \Gamma(E;R) \longrightarrow \Gamma(E;R)$ is clearly an isomorphism of R-modules. Since $q = q \circ m$, by definition of s, we have $\mu(s) = -s$ and, if $t \in \Gamma(X;R)$, we have $\mu(\epsilon(t)) = \epsilon(t)$. Therefore, $u(\epsilon(t))s = \epsilon(t) = \mu(\epsilon(t)) = -u(\epsilon(t))s$. Since, by Proposition 3.24, $\Gamma(E;R) \simeq R$ and R has no zero divisors, we have $u(\epsilon(t)) = -u(\epsilon(t))$ for all $t \in \Gamma(X;R)$. Thus $u \circ \epsilon : \Gamma(X;R) \longrightarrow \{r \in R : 2r = 0\}$ is an isomorphism as required. □

3.4 The fundamental class

We show below that an R-orientation of X determines in a canonical way a homology class called the fundamental class. The classical analogue of this theory is treated in [d] Chapter VIII.

Definition 3.26 Let A be a nonempty definably compact definable subset of X. By Corollary 2.4, let $\{\emptyset, A_1, \ldots, A_l\}$ be a finite family closed under intersection of definably compact definable subsets of A such that $A = \cup\{A_i : i = 1, \ldots, l\}$ and let $\{(V_1, U_1), \ldots, (V_k, U_k)\}$ be finitely many special pairs of open definable subsets of X such that for each i there is a j_i such that $A_i \subseteq U_{j_i}$.

A triple of the form (A_i, U_{j_i}, V_{j_i}) will be called a *special triple of* A.

Lemma 3.27 *Suppose that* A *is a nonempty definably compact definable subset of* X *and* (C, W, V) *a special triple of* A. *Then for every* $c \in C$, *the homomorphism*

$$j_c^C : H_n(X, X - C; R) \longrightarrow H_n(X, X - c; R)$$

is an isomorphism.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccc} H_n(X, X - C; R) & \simeq & H_n(V, V - C; R) & \simeq & \widetilde{H}_{n-1}(V - C; R) \\ \downarrow^{j_y^C} & & \downarrow & & \downarrow \\ H_n(X, X - y; R) & \simeq & H_n(V, V - y; R) & \simeq & \widetilde{H}_{n-1}(V - y; R) \end{array}$$

in which the left horizontal isomorphism are excisions and the right ones are connecting homomorphisms (V is acyclic). If we show that $V - C$ has the same o-minimal homology as the $(n-1)$-sphere, then by Remark 3.1 the right vertical arrow is an isomorphism. Therefore, j_y^C is an isomorphism.

By Corollary 2.4 (3), there is a definable triangulation (Ψ, M) of V such that $\Psi(C)$ is closure of the geometric realization of a simplex of M. Let $h : V \longrightarrow B_n(0, \epsilon)$ be a definable homeomorphism such that $W = h^{-1}(B_n(0, \delta))$ for some $0 < \delta < \epsilon$. Then the o-minimal homology of $V - C$ is the same as the o-minimal homology of $B_n(0, \epsilon) - h(C)$. Furthermore, $(\Psi \circ h^{-1}, M)$ is a definable triangulation of $B_n(0, \epsilon)$ such that $\Psi(h(C))$ is closure of the geometric realization of a simplex of M. Thus the result follows once we prove the following claim.

Claim 3.28 *Let* (Ψ, M) *be a definable triangulation of* $B_n(0, \epsilon) \subseteq N^n$ *and* r *an open simplex of* M *such that* $\overline{|r|} \subseteq |M|$ *and* $\Psi^{-1}(\overline{|r|}) \subseteq B_n(0, \epsilon)$. *Then for all* $q \in \mathbb{Z}$, *the group* $\widetilde{H}_q(B_n(0, \epsilon) - \Psi^{-1}(\overline{|r|}))$ *is* $\mathbb{Z}$ *for* $q = n - 1$ *and is zero otherwise.*

Proof. Take a barycentric subdivision of M and let $a \in |r|$ be the barycentre of r. Let $A = B_n(0, \epsilon) - \Psi^{-1}(a)$ and let $B = B_n(0, \epsilon) - \Psi^{-1}(\overline{|r|})$.

For each point x in the boundary $\partial\overline{|r|}$ of $\overline{|r|}$ in $|M|$ let $l_x^+(t)$ with $t > 0$ be the half line that starts from a and passes through x. The family $\{l_x^+(t) : x \in \partial\overline{|r|}\}$ is a uniformly definable family. Since by invariance of domain (see [Wo]) $|M|$ is open and $\overline{|r|}$ is definably compact, there is an open simplex s in N^n with barycentre a such that: (i) $\overline{|r|} \subseteq |s|$; (ii) every face of $\overline{s}$ in N^n is in an affine space parallel to the affine space determined by a face of $\overline{r} \subseteq M$ and (iii) $\overline{|s|} \subseteq |M|$. For each $x \in \partial\overline{|r|}$, let $s_x > 0$ be the unique element such that $l_x^+(s_x) \in \partial\overline{|s|}$ and let $r_x > 0$ be the unique element such that $l_x^+(r_x) \in \partial\overline{|r|}$ (i.e, such that $l_x^+(r_x) = x$). Let $C = B_n(0, \epsilon) - \Psi^{-1}(|s|)$ and let $i_C : C \longrightarrow B$ be the inclusion map. Let $f : |M| - \overline{|r|} \longrightarrow |M| - |s|$ be the definable continuous map given by $f(l_x^+(t)) = s_x$ for $r_x < t \leq s_x$, and $f(x) = x$ for all $x \in |M| - |s|$. Note that this map is in fact a well defined definable map since $\overline{|s|} - \overline{|r|}$ is the set $\{l_x^+(t) : r_x < t \leq s_x,\ x \in \partial\overline{|r|}\}$. Now let $F = \Psi \circ f \circ \Psi^{-1}$. Then we have $F \circ i_C = 1_C$ and $i_C \circ F$ is definably homotopic to 1_B via the definable homotopy $K : [0,1] \times B \longrightarrow B$ given by $K(z, b) = b$ for $b \in C$, and

$$K(z, \Psi^{-1}(l_x^+(t))) = \Psi^{-1}(l_x^+(\frac{s_x - (z(s_x - r_x) + r_x))}{s_x - r_x}(t - r_x) + z(s_x - r_x) + r_x))$$

on $B - C$. Hence, $F : B \longrightarrow C$ is a definable deformation retract and we have $\widetilde{H}_q(B) \simeq \widetilde{H}_q(C)$ for all $q \in \mathbb{Z}$.

A similar argument shows that there is a definable deformation retract $G : A \longrightarrow C$. Therefore we have $\widetilde{H}_q(C) \simeq \widetilde{H}_q(A) \simeq \widetilde{H}_q(S^{n-1})$ for all $q \in \mathbb{Z}$, where S^{n-1} is the $(n-1)$-sphere. But with this we end the proof of the claim. □

□

Note that Claim 3.28 follows from Case 1 in the proof of [bo2] Theorem 5.2 but we included here a different direct proof.

The classical analogue of the next theorem can be found in [d] Chapter VIII, Section 3. The use of Corollary 2.4 simplifies the argument.

Theorem 3.29 *Suppose that $A \subseteq X$ is a nonempty definably compact definable subset of X. Then*

$$j_A : H_n(X, X - A; R) \longrightarrow \Gamma(A; R)$$

is an isomorphism.

Proof. We first show that if the result holds for nonempty definably compact definable subsets A, B and $A \cap B$ then it holds for $C = A \cup B$.

In fact, using the Mayer-Vietoris sequence for the excisive triad $(X - A \cap B; X - A, X - B)$ we get $H_q(X, X - C; R) = 0$ for $q > n$ and we have the commutative diagram

$$\begin{array}{ccccc} H_n(X, C'; R) & \stackrel{(j_{1*}, -j_{2*})}{\longrightarrow} & H_n(X, A'; R) \oplus H_n(X, B'; R) & \stackrel{i_{1*}+i_{2*}}{\longrightarrow} & H_n(X, D'; R) \\ \downarrow j_C & & \downarrow j_A \oplus j_B & & \downarrow j_D \\ 0 \longrightarrow \Gamma(C; R) & \stackrel{(r^C_A, -r^C_B)}{\longrightarrow} & \Gamma(A; R) \oplus \Gamma(B; R) & \stackrel{r^A_D + r^B_D}{\longrightarrow} & \Gamma(D; R) \end{array}$$

where $D = A \cap B$, $C' = X - C$, $A' = X - A$, $B' = X - B$ and $D' = X - D$. The 5-lemma implies that j_C is an isomorphism.

We now apply Corollary 2.4. With out loss of generality assume that A is definably connected and let $\{\emptyset, A_1, \ldots, A_l\}$ be a finite family closed under intersection of definably compact definable subsets of A such that $A = \cup\{A_i : i = 1, \ldots, l\}$ and let $\{(V_1, U_1), \ldots, (V_k, U_k)\}$ be finitely many special open definable subsets such that for each i there is a j_i such that $A_i \subseteq U_{j_i}$. By what we saw above, the theorem follows by induction on l. So suppose that $C = A_i$ and let $(V, W) = (V_{j_i}, U_{j_i})$.

Note that, since $j^W_x = j^C_x \circ j^W_C$ for all $x \in C$, by Lemmas 3.4 and 3.27, j^W_C is an isomorphism. Let $s \in \Gamma(C; R)$ and $c \in C$. By Lemma 3.4, there is a unique $\alpha_W \in H_n(X, X - W; R)$ such that $s(c) = (c, j^W_c(\alpha_W))$. Let $r = j_W(\alpha_W)_{|C}$. Then by Proposition 3.12, $r \in \Gamma(C; R)$. Since $r(c) = s(c)$, by Lemma 3.10, we have $r = s$. Since $r = j_C(j^W_C(\alpha_W))$ and $j^W_C(\alpha_W)$ is uniquely determined, the result holds. $\square$

Remark 3.13, Theorem 3.29 and the 5-lemma imply the following.

Remark 3.30 Suppose that $B \subseteq A \subseteq X$ are nonempty definably compact definable subsets of X. Then

$$j_{A,B} : H_n(X - B, X - A; R) \longrightarrow \Gamma(A, B; R)$$

is an isomorphism.

Corollary 3.31 *Suppose that X is definably compact. Then $s \in \Gamma(X; R)$ is an R-orientation of X if and only if there is a unique element ζ_X of $H_n(X; R)$ such that $s = j_X(\zeta_X)$. In particular, the notion of R-orientation of a definably compact definable manifold X is independent of models.*

Proof. The first part of the theorem follows at once from Theorem 3.29. The rest follows from the invariance of $H_n(X;R)$ in models (see for example [bo2]). $\Box$

Definition 3.32 The element ζ_X of $H_n(X;R)$ from Corollary 3.31 is called the *fundamental class* of the R-orientation of X.

The *orientation class of* X is the element $\omega_X \in H^n(X;R)$ such that $(\omega_X, \zeta_X) = 1$.

Remark 3.33 Let X and Y be definably compact definable manifolds, $s \in \Gamma(X;R)$ an R-orientation of X and $t \in \Gamma(Y;R)$ an R-orientation of Y. It is easy to see that, if ζ_X and ζ_Y are the corresponding fundamental classes, then $\zeta_X \times \zeta_Y$ is the fundamental class of the R-orientation $\gamma(s,t)$ of $X \times Y$ given by Remark 3.20.

The argument in the proof of the next result is the same as in [d] Chapter VIII, Corollary 3.4 except that we use Lemma 3.4, Proposition 3.22 instead of their topological analogues.

Corollary 3.34 *Suppose that X is definably compact and definably connected. Assume that R has no zero divisors. Then $H_n(X;R) = R$ if X is R-orientable and $H_n(X;R) = \{r \in R : 2r = 0\}$ otherwise.*

Proof. If X is R-orientable, apply Theorem 3.29 and Proposition 3.22. For the converse, apply Theorem 3.29 and Proposition 3.25. $\Box$

We end this subsection with the construction of relative fundamental classes.

Definition 3.35 Suppose that A is a nonempty definable subset of X. An *R-orientation of X along A* is a section $s \in \Gamma(A;R)$ such that for each $a \in A$, if $s(a) = (a, s'(a))$, then $s'(a)$ is an R-orientation of X at a.

By Theorem 3.29, if A is definably compact, given an R-orientation $s \in \Gamma(A;R)$ of X along A there is a unique element $\zeta_{X,A} \in H_n(X, X - A; R)$ such that $s = j_A(\zeta_{X,A})$. We call $\zeta_{X,A}$ the *(relative) fundamental class* of the R-orientation of X along A. We sometimes write ζ_A for $\zeta_{X,A}$.

The following results can be obtained by adapting the proofs of Propositions 3.22, 3.24 and 3.25.

Remark 3.36 Let A be a nonempty definable subset of X and suppose that R has no zero divisors. Then the following hold.

(1) If A has k definably connected components and X is R-orientable along A, then $\Gamma(A; R) \simeq R^k$.

(2) If A is definably connected, definably compact and X is non R-orientable along A, then $\Gamma(A; R) \simeq \{r \in R : 2r = 0\}$.

3.5 Sections with definably compact supports

For a nonempty closed definable subset A of X and $s \in \Gamma(A; R)$, the *support of s* is by definition the subset of A where s does not coincide with the zero section. By Lemma 3.10, the support of a section of s is a clopen definable subset of A.

Let $\Gamma_c(A; R)$ the subset of $\Gamma(A; R)$ of all sections whose support is a definably compact definable subset of A.

Lemma 3.37 *Suppose that A is a nonempty closed definable subset of X. Then $\Gamma_c(A; R)$ is an R-submodule of $\Gamma(A; R)$ such that:*

(i) if A is definably compact, then $\Gamma_c(A; R) = \Gamma(A; R)$;

(ii) if A is definably connected and not definably compact, then $\Gamma_c(A; R) = 0$.

Proof. Let $s, t \in \Gamma_c(A; R)$ and suppose that the support of s (resp., t) is the definable subset S (resp., T) of A. Let $S' = \{x \in S : s(x) = -t(x)\}$ and $T' = \{x \in T : s(x) = -t(x)\}$. By Lemma 3.10, S' (resp., T') is a clopen definable subset of S (resp., T). But $s + t$ has support $(S - S') \cup (T - T')$ and so, $s + t \in \Gamma_c(A; R)$. On the other hand, for $r \in R$, rs has support S (resp., $\emptyset$) when $r \neq 0$ (resp., $r = 0$). So $rs \in \Gamma_c(A; R)$.

Finally, (i) and (ii) follow immediately from Lemma 3.10. □

Lemma 3.38 *Let A be a nonempty closed definable subset of X. Then the homomorphism $j_A : H_n(X, X - A; R) \longrightarrow \Gamma(A; R)$ has image in $\Gamma_c(A; R)$.*

Proof. Let $\alpha \in H_n(X, X-A; R)$ and let $z = \sum_{i=1}^{k} r_i z_i \in S_n(X;R)$ with $\partial z \in S_{n-1}(X-A;R)$ be a representative of α. Let $S = \cup\{z_i(\Delta^n) : i = 1, \dots, k\}$. By [vdd] Chapter VI, Proposition 1.10, S is a definably compact definable subset of X. Clearly the support of $s = j_A(\alpha) \in \Gamma(A;R)$ is $S \cap A$ and hence $s \in \Gamma_c(A;R)$. $\Box$

The classical analogues of the next theorem can be found in [d] Chapter VIII, Section 3 and in [g] Chapter 22. Our proof follows closely that of [g] Theorem 22.24, but the use of Corollary 2.4 and Theorem 3.29 simplifies the argument.

Theorem 3.39 *Suppose that $A \subseteq X$ is a closed nonempty definable subset of X. Then*

$$j_A : H_n(X, X-A; R) \longrightarrow \Gamma_c(A;R)$$

is an isomorphism.

Proof. By Theorem 3.29 this holds if A is definably compact. Furthermore, arguing as in the proof of Theorem 3.29, we see that if this result holds for closed nonempty definable subsets A, B and $A \cap B$ then it holds for $C = A \cup B$.

Claim (1): If $A \subseteq U$ where U is an open definable subset of X with definably compact closure $\overline{U}$, then the result holds for U and A.

Proof of Claim (1): Let $W = (U - A) \cup (X - \overline{U})$ and $V = U \cup (X - \overline{U})$. We use the exact homology sequence for the triple (X, V, W).

Note that by excision, $H_q(U, U-A; R) \simeq H_q(V, W); R)$. For $q > n$ we have $\cdots \longrightarrow H_{q+1}(X, V); R) \longrightarrow H_q(U, U-A; R) \longrightarrow H_q(X, W; R) \longrightarrow \cdots$.

For $q = n$, we have

$$\begin{array}{ccccccc} 0 \longrightarrow & H_n(U, U-A; R) & \longrightarrow & H_n(X, W; R) & \longrightarrow & H_n(X, V; R) \\ & \downarrow j_A^U & & \downarrow & & \downarrow \\ 0 \longrightarrow & \Gamma_c(A;R) & \xrightarrow{i} & \Gamma(A \cup (\overline{U} - U); R) & \xrightarrow{r} & \Gamma(\overline{U} - U; R) \end{array}$$

where $\Gamma_c(A;R)$ and j_A are computed in the definable manifold U, and the monomorphism i is defined as follows: if $s \in \Gamma_c(A;R)$ is zero outside a definably compact subset K of A, then $i(s)_{|A} = s$ and $i(s) = 0$ outside K.

Applying the result for the definably compact definable sets $\overline{U} - U$ and $A \cup (\overline{U} - U)$, we see that $H_q(U, U - A; R) = 0$ for $q > n$ and j_A^U is an isomorphism. $\square$

Claim (2): The result holds for any closed nonempty definable subset A.

Proof of Claim (2): Let $s \in \Gamma_c(A; R)$ and suppose that s is zero outside the definably compact subset K of A. Then by Corollary 2.4, there is an open definable subset $K \subseteq U$ such that $\overline{U}$ is definably compact.

Consider $A' = A \cap U$ and $s' = s_{|A'}$. By Claim (1) applied to U and A', and the commutative diagram

$$\begin{array}{ccc} H_n(U, U - A'; R) & \longrightarrow & H_n(X, X - A; R) \\ \downarrow j_{A'} & & \downarrow j_A \\ 0 \longrightarrow s' \in \Gamma_c(A'; R) & \stackrel{i}{\rightarrow} & s \in \Gamma_c(A; R) \end{array}$$

we see that j_A is surjective.

Now let $\alpha \in H_q(X, X - A; R)$ and let $z \in S_q(X; R)$ with $\partial z \in S_{q-1}(X - A; R)$ be the relative cycle representing α. If $q = n$, suppose that $j_A(\alpha) = 0$. Applying the above argument to $\mathrm{Im} z$ (the support of z, see [Wo]), there is an open definable subset U of X such that $\mathrm{Im} z \subseteq U$ and $\overline{U}$ is definably compact. Let $A' = A \cap U$. By the same commutative diagram, we have $\alpha = 0$. For $q > n$, we know that the class of z in $H_q(U, U - A'; R)$ is zero by Claim (1), so $\alpha = 0$. $\square$

Remark 3.40 Let $B \subseteq A \subseteq X$ be nonempty closed definable subsets of X. Let $\Gamma_c(A, B; R)$ be the kernel of the restriction homomorphism $r_B^A : \Gamma_c(A; R) \longrightarrow \Gamma_c(B; R)$. Theorem 3.39, the 5-lemma and the commutative diagram

$$\begin{array}{ccccccc} & & H_n(X - B, X - A; R) & \rightarrow & H_n(X, X - A; R) & \rightarrow & H_n(X, X - B; R) \\ & & \downarrow j_{A,B} & & \downarrow j_A & & \downarrow j_B \\ 0 & \rightarrow & \Gamma_c(A, B; R) & \rightarrow & \Gamma_c(A; R) & \rightarrow & \Gamma_c(B; R) \end{array}$$

imply that $j_{A,B} : H_n(X - B, X - A; R) \longrightarrow \Gamma_c(A, B; R)$ is an isomorphism.

Corollary 3.41 *If A is a closed, nonempty, definably connected and not definably compact definable subset of X, then $H_n(X, X - A; R) = 0$. In particular, if X is definably connected and not definably compact then $H_n(X; R) = 0$.*

4 Sections with bounded support

Our goal here is to prove a version of Theorem 3.39 for nonempty definable subsets A of X such that $X - A$ is definably locally closed (e.g., $X - A$ is open or closed). In order to do this we first have to prove some results about definably locally closed sets.

4.1 Definably locally closed sets

In this subsection we present the basic properties of definably locally compact definable sets that will be useful later. This is the definable analogue of the theory of [d] Chapter IV, Section 8.

Proposition 4.1 *Let $Z \subseteq N^n$ be a definable subset. Then the following are equivalent:*

(1) Z is of the form $C \cap U$ where C is a closed definable subset of N^n and U is an open definable subset of N^n.

(2) There is a definable retraction $r : V \longrightarrow Z$ where V is an open definable subset of N^n.

(3) There is a definable family $\{U_z : z \in Z\}$ of definable open subsets of N^n such that for every $z \in Z$ we have $z \in U_z$ and $U_z \cap Z$ is closed in U_z.

(4) There is a definable family $\{V_z : z \in Z\}$ of definable subsets of N^n such that for every $z \in Z$ the set V_z is a definably compact definable neighbourhood of z in Z.

Proof. If (1) holds then Z is closed in U and (2) follows from [vdd] Chapter VIII, Proposition 3.3. If (2) holds, then regarding r as a definable map $r : V \longrightarrow V$, we have $Z = \overline{Z} \cap V$ since $Z = \overline{Z} = \{v \in V : r(v) = v\}$. Thus (1) holds.

Assume (4). Then $V_z = Z \cap W_z$ where $\{W_z : z \in Z\}$ is a definable family of open definable subsets of N^n. For $z \in Z$, let $U_z = \mathring{W}_z$. Then $Z \cap U_z = V_z \cap U_z$ is closed in U_z. Therefore, the definable family $\{U_z : z \in Z\}$ satisfies (3).

If (3) holds, then we have $Z \cap U_z = \overline{Z} \cap U_z$, hence $Z = Z \cap (\cup\{U_z : z \in Z\}) = \cup\{\overline{Z} \cap U_z : z \in Z\} = \overline{Z} \cap (\cup\{U_z : z \in Z\})$. This implies (1) since $U = \cup\{U_z : z \in Z\}$ is an open definable subset of N^n.

Finally we show that (1) implies (4). Let $\{B_z : z \in Z\}$ be a family of definably compact definable neighbouhoods of $z \in Z$ in N^n. For $z \in Z$, let $W_z = Z \cap B_z = C \cap B_z$. Then each W_z is definably compact and the definable family $\{W_z : z \in Z\}$ shows (4). $\square$

A definable set $Z \subseteq N^n$ satisfying one the conditions of Proposition 4.1 will be called *definably locally closed* or *definably locally compact*.

The next remark is standard. For the semi-algebraic analogue, see [BCR] Proposition 2.2.9.

Remark 4.2 Every definably locally closed definable set $Z \subseteq N^n$ is definably homeomorphic to a closed definable subset of N^{n+1}.

In fact, let C and U be closed and open definable subsets of N^n such that $Z = C \cap U$. Then the definable map $h : U \longrightarrow N^n \times N$ given by $h(x) = (x, \frac{1}{d(x, N^n - U)})$ where $d(x, N^n - U) = \inf\{|x - y| : y \in N^n - U\}$, is a definable embedding of U into N^{n+1} (the projection $\pi : N^{n+1} \longrightarrow N$ onto the first n-coordinates is the inverse of h). Moreover, $h(U) = \{(x, y) \in N^n \times N : y \cdot d(x, N^n - U) = 1\}$ is a closed definable subset of N^{n+1}. Hence, $h(Z)$ is closed in $h(U)$ and hence in N^{n+1}.

Proposition 4.3 *Let Z be a definable subset of N^n which is definably locally closed. If $f, g : Y \longrightarrow Z$ are definable continuous maps and B is a definable subset of Y such that $f_{|B} = g_{|B}$, then there is an open definable neighbourhood U of B in Y and a definable homotopy $F : U \times [0,1] \longrightarrow X$ between $f_{|U}$ and $g_{|U}$ such that for all $x \in B$ and $t \in [0,1]$ we have $F(x,t) = f(x)$.*

Proof. By Proposition 4.1, there is a definable retraction $r : V \longrightarrow Z$ where V is an open definable subset of N^n. Let $i : Z \longrightarrow V$ be the inclusion and let $U = \{y \in Y : \{(1-t)i \circ f(y) + ti \circ g(y) : t \in [0,1]\} \subseteq V\}$. Then U is an open definable neighbourhood of B in Y. Define $F : U \times [0,1] \longrightarrow X$ by $F(y,t) = r((1-t)i \circ f(y) + ti \circ g(y))$. $\square$

Corollary 4.4 *Let $B \subseteq Z$ be a definable subsets of N^n which are definably locally closed. Then there is a definable neighbourhood retract $r : V \longrightarrow B$ in*

Z and there is an open definable neighbourhood W of B in V such that $i \circ r_{|W}$ is definably homotopic to the inclusion $j : W \longrightarrow V$, where $i : B \longrightarrow V$ is the inclusion.

Proof. By Proposition 4.1 one can assume that V is open in N^n. Hence V is definably locally closed. Now apply Proposition 4.3 with $f = i \circ r$ and $g = 1_V$. □

4.2 Sections with bounded support

Given a nonempty definable subset A of X, we denote by $\Gamma_b(A;R)$ the subset of $\Gamma(A;R)$ of all sections $s \in \Gamma(A;R)$ such that the closure in X of the support of s is a definably compact definable subset of X. For nonempty definable subsets $B \subseteq A$ of X, we denote by $\Gamma_b(A,B;R)$ the kernel of the restriction homomorphism $r^A_B : \Gamma_b(A;R) \longrightarrow \Gamma_b(B;R)$.

Lemma 4.5 *Suppose that A is a nonempty definable subset of X. Then $\Gamma_b(A;R)$ is an R-submodule of $\Gamma(A;R)$ such that:*

(i) if $\overline{A}$ is definably compact, then $\Gamma_b(A;R) = \Gamma(A;R)$;

(ii) if A is definably connected but $\overline{A}$ is not definably compact, then we have $\Gamma_b(A;R) = 0$.

Proof. The fact that $\Gamma_b(A;R)$ is an R-submodule of $\Gamma(A;R)$ can be proved as in Lemma 3.37. Finally, (i) is obvious and for (ii) let $s \in \Gamma_b(A;R)$ and suppose that the closure $\overline{S}$ in X of the support S of s is a definably compact definable subset of X. It is enough to show that $S = \emptyset$. By Lemma 3.10, $S = \emptyset$ or $S = A$. But the later cannot happen since $\overline{A}$ is not definably compact. □

Lemma 4.6 *Let A be a nonempty definable subset of X. Then the homomorphism $j_A : H_n(X, X - A;R) \longrightarrow \Gamma(A;R)$ has image in $\Gamma_b(A;R)$.*

Proof. Let $\alpha \in H_n(X, X - A;R)$ and let $z = \sum_{i=1}^{k} r_i z_i \in S_n(X;R)$ with $\partial z \in S_{n-1}(X - A;R)$ be a representative of α. Let $S = \cup\{z_i(\Delta^n) : i =$

$1, \dots, k\}$. By [vdd] Chapter VI, Proposition 1.10, S is a definably compact definable subset of X. Clearly the support of $s = j_A(\alpha) \in \Gamma(A; R)$ is $S \cap A$ and hence $s \in \Gamma_b(A; R)$. $\square$

For the proof of the main result of this subsection the following lemma will be very crucial.

Lemma 4.7 *Suppose that X is definably connected and A is a nonempty definable subset of X. Then $\Gamma_b(X, A; R) = 0$, the homomorphism $H_n(X - A; R) \longrightarrow H_n(X; R)$ induced by the inclusion is the zero homomorphism and $H_{q-1}(X - A; R) \simeq H_q(X, X - A; R)$ for all $q > n$. Moreover, we have the following commutative diagram*

$$\begin{array}{ccccccccc} 0 \to & H_n(X;R) & \xrightarrow{k_*} & H_n(X, X-A;R) & \to & \widehat{H}_{n-1}(X-A;R) & \to 0 \\ & \downarrow^{j_X} & & \downarrow^{j_A} & & \downarrow^{\widehat{j}_A} & \\ 0 \to & \Gamma_b(X;R) & \xrightarrow{k'} & \Gamma_b(A;R) & \to & \widehat{\Gamma}(A;R) & \to 0 \end{array}$$

where $j_X : H_n(X; R) \longrightarrow \Gamma_b(X; R)$ is an isomorphism, $\widehat{H}_{n-1}(X - A; R) = \ker(H_{n-1}(X - A; R) \longrightarrow H_{n-1}(X; R)) \simeq \mathrm{coker}(k_*)$, $\widehat{\Gamma}(A; R) = \mathrm{coker}(k')$ and $\widehat{j}_A$ is induced by j_A.

Proof. By Lemma 3.10, the set on which two sections in $\Gamma(X; R)$ agree is a clopen definable subset of X. Hence, $A \neq \emptyset$ and X definably connected implies that $\Gamma(X, A; R) = \Gamma_b(X, A; R) = 0$.

The fact that $H_n(X - A; R) \longrightarrow H_n(X; R)$ is the zero homomorphism is proved in the following way: if X is not definably compact, then by Corollary 3.41 we have $H_n(X; R) = 0$; if X is definably compact, use the diagram of Remark 3.13 applied to the definable subsets $A \subseteq X$ of X, the fact that $\Gamma(X, A; R) = 0$ (see the paragraph above) together with Theorem 3.29. Hence, the exactness axiom shows that $H_{q-1}(X - A; R) \simeq H_q(X, X - A; R)$ for all $q > n$.

Thus $H_n(X - A; R) \simeq H_{n+1}(X, X - A; R) = 0$ and, as we showed in the two paragraphs above, we have that $j_X : H_n(X; R) \longrightarrow \Gamma_b(X; R)$ is an isomorphim. Thus by Lemma 4.6 we obtain the diagram of the lemma. $\square$

We are now ready to generalize Theorem 3.39. Our proof follows closely Dold's proof (see [d] Chapter VIII, Proposition 3.3) of the corresponding classical result. The main diference is that we use the o-minimal versions of the classical facts used by Dold and we use Theorem 3.29 to simplify the proof.

Theorem 4.8 *Let $A \subseteq X$ be a nonempty definable subset of X. If $X - A$ is definably locally closed, then*

$$j_A : H_n(X, X - A; R) \longrightarrow \Gamma_b(A; R)$$

is an isomorphism.

Proof. Since clearly X is definably locally closed (apply Proposition 4.1 (4)), by Remark 4.2, there is always k such that X is closed in N^k. So we will assume that X is closed in N^k.

We may also assume that X is definably connected (otherwise, we prove the theorem for each definably connected component of X).

Since $X - A$ is a definably locally closed and $X - A \neq X$, by Proposition 4.1, there is a definable open proper subset U of X and a definable retraction $r : U \longrightarrow X - A$. By Corollary 4.4, we may assume that $i^{X-A} \circ r$ is definably homotopic to the inclusion $i^U : U \longrightarrow X$, where $i^{X-A} : X - A \longrightarrow X$ is the inclusion.

On the other hand, we have $i_*^{X-A} \circ r_* = i_*^U$. Thus, r_* maps $\widehat{H}_{n-1}(U; R) = \ker i_*^U$ into $\widehat{H}_{n-1}(X - A; R) = \ker i_*^{X-A}$ and is a left inverse to the induced homomorphism $i_* : \widehat{H}_{n-1}(X - A; R) \longrightarrow \widehat{H}_{n-1}(U; R)$. So i_* is monomorphic and the diagram

$$\begin{array}{ccc} \widehat{H}_{n-1}(X - A; R) & \xrightarrow{i_*} & \widehat{H}_{n-1}(U; R) \\ \downarrow \widehat{j}_A & & \downarrow \widehat{j}_{X-U} \\ \widehat{\Gamma}(A; R) & \rightarrow & \widehat{\Gamma}(X - U; R) \end{array}$$

shows that $\widehat{j}_A$ is monomorphic since, by Lemma 4.7 and Theorem 3.29, $\widehat{j}_{X-U}$ is an isomorphism. We will show that $\widehat{j}_A$ is also epimorphic and hence, by Lemma 4.7, j_A is an isomorphism as required.

By Corollary 4.4, for each $x \in A$, choose a definable open subset V_x such that $X - A \subseteq V_x \subseteq U - x$ and $i_x \circ r_{|V_x}$ is definably homotopic to the inclusion $k_x : V_x \longrightarrow U - x$, where $i_x : X - A \longrightarrow U - x$ is the inclusion. Then

$i_{x*} \circ (r_{|V_x})_* = k_{x*}$ and we have the following commutative diagram

$$\begin{array}{ccccccc}
\widehat{H}_{n-1}(A';R) & \stackrel{j_{x*}}{\rightarrow} & \widehat{H}_{n-1}(V_x;R) & \stackrel{k_{x*}}{\rightarrow} & \widehat{H}_{n-1}(U_x;R) & \stackrel{l_{x*}}{\rightarrow} & \widehat{H}_{n-1}(U;R) \\
\downarrow\widehat{j}_A & & \downarrow\widehat{j}_{V'_x} & & \downarrow\widehat{j}_{U'_x} & & \downarrow\widehat{j}_{U'} \\
\widehat{\Gamma}(A;R) & \stackrel{j'_{x*}}{\rightarrow} & \widehat{\Gamma}(V'_x;R) & \stackrel{k'_{x*}}{\rightarrow} & \widehat{\Gamma}(U'_x;R) & \stackrel{l'_{x*}}{\rightarrow} & \widehat{\Gamma}(U';R)
\end{array}$$

where $U_x = U - x$ and for $W \in \{A, V_x, U_x, U\}$, we use W' to denote $X - W$.

We have $i_{x*} = k_{x*} \circ j_{x*}$ and we define $i'_x = k'_x \circ j'_x$ and $i' = l_x \circ i'_x$. Moreover, if we define $\rho_x = (r_{|V_x})_* \circ (\widehat{j}_{V'_x})^{-1} \circ j'_x : \widehat{\Gamma}(A;R) \longrightarrow \widehat{H}_{n-1}(A';R)$ (we use here the fact that, by Lemma 4.7 and Theorem 3.29, $\widehat{j}_{V'_x}$ is an isomorphism), then a simple computation using the diagram above shows that $i'_x \circ \widehat{j}_A \circ \rho_x = i'_x$. Composing this equality with l'_x we get $(i' \circ \widehat{j}_A) \circ \rho_x = i'$. Note that the right side of this equality does not depend on x and $i' \circ \widehat{j}_A = \widehat{j}_{U'} \circ i_*$ is monomorphic. Hence, $\rho = \rho_x$ is independent of x. We will show that $\widehat{j}_A \circ \rho = 1_{\widehat{\Gamma}(A;R)}$ and so $\widehat{j}_A$ is epimorphic.

Let $s \in \Gamma_b(A;R)$ and let $\widehat{s} \in \widehat{\Gamma}(A;R)$ be its coset. Let $\sigma \in \Gamma_b(A;R)$ be a representative of $\widehat{j}_A \circ \rho(\widehat{s})$. Then $i'_x \circ \widehat{j}_A \circ \rho(\widehat{s}) = i'_x(\widehat{s})$. Therefore, there is $t_x \in \Gamma_b(X;R)$ such that $s_{|U'_x} - \sigma_{|U'_x} = t_{x|U'_x}$. In particular, we have $s_{|U'} - \sigma_{|U'} = t_{x|U'}$ and so, by Lemma 4.7, $t = t_x$ is independent of x. Thus, $s(x) - \sigma(x) = t(x)$ for all $x \in A$ as required. □

Remark 4.9 For nonempty definable subsets $B \subseteq A$ of X, we denote by $\Gamma_b(A, B;R)$ the kernel of the restriction homomorphism $r^A_B : \Gamma_b(A;R) \longrightarrow \Gamma_b(B;R)$. The 5-lemma, Theorem 4.8 and the commutative diagram

$$\begin{array}{ccccccc}
 & & H_n(X-B, X-A;R) & \rightarrow & H_n(X, X-A;R) & \rightarrow & H_n(X, X-B;R) \\
 & & \downarrow j_{A,B} & & \downarrow j_A & & \downarrow j_B \\
0 & \rightarrow & \Gamma_b(A,B;R) & \rightarrow & \Gamma_b(A;R) & \rightarrow & \Gamma_b(B;R)
\end{array}$$

imply that $j_{A,B} : H_n(X-B, X-A;R) \longrightarrow \Gamma_b(A,B;R)$ is an isomorphism when $X - A$ and $X - B$ are definably locally closed.

Theorem 4.8 implies at once the following observation.

Corollary 4.10 *If X is definably compact, then the morphism $j : O^X \longrightarrow \widetilde{O}^X$ of presheafs is an isomorphism.*

For a definable subset A of X, let $c_b(A)$ be the number of definably connected components of A whose closure in X is definably compact.

Corollary 4.11 *Let A be a definable subset of X such that $X - A$ is definably locally closed and A intersects every definably connected component of X. If X is R-orientable, then*

$$c_b(A) = c_b(X) + \dim(\ker[H_{n-1}(X - A; R) \longrightarrow H_{n-1}(X; R)]).$$

Proof. We start the proof with the following two claims.

Claim 4.12 *Let A be a nonempty definable subset of X. If X is R-orientable along A, then $c_b(A) = \dim \Gamma_b(A; R)$.*

Proof. Let $A_1, \ldots, A_l$ be the definably connected components of A. Then $\Gamma_b(A; R) = \oplus_{i=1}^{l} \Gamma_b(A_i; R)$. So we may assume that A is definably connected. If $\overline{A}$ is not definably compact, then $c_b(A) = 0$ and by Lemma 4.5 we also have $\Gamma_b(A; R) = 0$. If $\overline{A}$ is definably compact, then $c_b(A) = 1$ and we must show that $\dim \Gamma_b(A; R) = 1$, i.e., $\dim \Gamma(A; R) = 1$. This is a consequence of the Remark 3.36 (2). □

Claim 4.13 *Let X be an R-orientable definable manifold of dimension n. Then $c_b(X) = \dim H_n(X; R)$.*

Proof. Let $X_1, \ldots, X_l$ be the definably connected components of X. Then $H_n(X; R) = \oplus_{i=1}^{l} H_n(X_i; R)$ and, by Remark 3.19, each X_i is an R-orientable definable manifold of dimension n. On the other hand, by Corollary 3.41, $H_n(X_i; R) = 0$ if X_i is not definably compact and by Corollary 3.34, $H_n(X_i; R) = R$ if X_i is definably compact. Therefore, $c_b(X) = \dim H_n(X; R)$. □

Thus by Claim 4.12 and Theorem 4.8, we have $c_b(A) = \dim H_n(X, X - A; R)$. On the other hand, by Claim 4.13, $c_b(X) = \dim H_n(X; R)$. Let $X_1, \ldots, X_l$ be the definably connected components of X and, for each $i = 1, \ldots, l$, let $A_i = X_i \cap A$. Then $X_i - A_i$ is definably locally closed in X_i and by Theorem 4.8, we have $H_n(X_i - A_i; R) \simeq \Gamma_b(X_i, A_i; R)$. Since, $H_n(X - A; R) =$

$\oplus_{i=1}^{l} H_n(X_i - A_i; R)$ and by Lemma 4.7, $\Gamma_b(X_i, A_i; R) = 0$, the corollary follows from the exact sequence

$$\to H_n(A'; R) \to H_n(X; R) \to H_n(X, A'; R) \to H_{n-1}(A'; R) \to H_{n-1}(X; R) \to$$

where $A' = X - A$. $\square$

If in Corollary 4.11 we put $X = S^m$ and $X - A$ definably homeomorphic to S^{m-1}, then we recover the o-minimal Jordan-Brouwer separation theorem from [Wo].

4.3 Definable ∂-manifolds

In the next remark we assume the readers familiarity with the theory of definable ∂-manifolds treated in [e].

Remark 4.14 Let X be a definably connected, definably compact definable ∂-manifold of dimension n. Let $\mathring{X}$ be the interior of X, ∂X the boundary of X and $\widehat{X}$ the definable manifold obtained from X by attaching a colar.

We say that X is R-orientable if $\mathring{X}$ is R-orientable (equivalently, if $\widehat{X}$ is R-orientable along $\mathring{X}$).

Consider the following commutative diagram

$$\begin{array}{ccc} H_n(\widehat{X}, \widehat{X} - \mathring{X}; R) & \stackrel{j_{\mathring{X}}}{\to} & \Gamma(\mathring{X}; R) \\ \uparrow & & \uparrow r^{X}_{\mathring{X}} \\ H_n(\widehat{X}, \widehat{X} - X; R) & \stackrel{j_X}{\to} & \Gamma(X; R) \end{array}$$

where the unmarked arrow is induced by inclusion. Since the inclusion determines isomorphisms $H_q(\widehat{X} - X; R) \longrightarrow H_q(\widehat{X} - \mathring{X}; R)$ for all $q \in \mathbb{Z}$, by the exactness axiom, the unmarked arrow is an isomorphism. By Theorem 3.29 (resp., 4.8), j_X (resp., $j_{\mathring{X}}$) is an isomorphism. Hence, X is R-orientable if and only if $\widehat{X}$ is R-orientable along X.

By Corollary 3.41, we have $H_n(\widehat{X}; R) = 0$. Hence, by the exactness axiom, the connecting homomorphism $\partial : H_n(\widehat{X}, \partial X; R) \longrightarrow H_{n-1}(\partial X; R)$ is injective. Finally note that, we also have an isomorphism $H_n(X, \partial X; R) \longrightarrow H_n(\widehat{X}, \widehat{X} - \mathring{X}; R)$ induced by the inclusion.

Therefore, if $s \in \Gamma(\mathring{X}; R)$ is an R-orientation of $\mathring{X}$, then the following hold:

(i) there is a unique homology class $\zeta \in H_n(X, \partial X; R)$ such that $s = j_{\mathring{X}}(\zeta'')$ where ζ'' is the image of ζ in $H_n(\widehat{X}, \widehat{X} - \mathring{X}; R)$;

(ii) there is a canonical R-orientation s_X of $\widehat{X}$ along X given by $s_X = j_X(\zeta')$, where $\zeta' \in H_n(\widehat{X}, \widehat{X} - X; R)$ is such that its image in $H_n(\widehat{X}, \widehat{X} - \mathring{X}; R)$ is ζ'';

(iii) there is a canonical R-orientation $\partial_X(s)$ of ∂X given by $\partial_X(s) = j_{\partial X}(\partial\zeta)$.

References

[BCR] J.Bochnak, M. Coste and M-F. Roy *Real Algebraic Geometry* Springer-Verlag 1998.

[bo1] A.Berarducci and M.Otero *Intersection theory for o-minimal manifolds* Ann. Pure and Appl. Logic 107 (2001) 87-119.

[bo2] A.Berarducci and M.Otero *Some examples of transfer methods for o-minimal structures* J. Symbolic Logic (3) 68 (2003) 785-794.

[D] H.Delfs *Homology of locally semialgebraic spaces* LNM 1484 Springer-Verlag 1991.

[d] A.Dold *Lectures on algebraic topology* Springer Verlag 1980.

[vdd] L. van den Dries *Tame Topology and o-minimal structures* Cambridge University Press 1998.

[e] M.Edmundo *Covering definable manifolds by open definable subsets* Preprint 2004.

[g] M.Greenberg *Lectures on algebraic topology* W.A.Benjamin 1967.

[pt] Y.Peterzil *Some topological and differentiable invariants in o-minimal structures* to appear in Proc. Ravello Meeting.

[ps] Y.Peterzil and C.Steinhorn *Definable compactness and definable subgroups of o-minimal groups* J. London Math. Soc. 59 (3) (1999) 769-786.

[Wo] A.Woerheide *O-minimal homology* PhD. Thesis (1996), University of Illinois at Urbana-Champaign.