On Pillay's conjecture for orientable definable groups

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Abstract

We prove here Pillay's conjecture [Type-definability, compact Lie groups and o-minimality, J. Math. Logic 4 (2) (2004) 147–162] for orientable, definably connected, definably compact definable groups in arbitrary o-minimal structures with definable Skolem functions.

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1 Introduction

Let $\mathcal{M} = (M, <, \dots)$ be a sufficiently saturated o-minimal structure with definable Skolem functions. By definable we will mean definable in $\mathcal{M}$ possibly with parameters.

A definable group is a group whose underlying set is a definable set and the graphs of the group operations are definable sets. The notion of definably compact is the analogue of the notion of semi-algebraically complete and was introduced by Peterzil and Steinhorn in [22]. A definable group is definably connected if it has no proper definable subgroups of finite index ([28]). For the basic theory of definable groups we refer the reader to [7], [24], [26], [27] and [28].

Here we prove the following theorem:

Theorem 1.1 *Let G be a $\mathbb{Z}/q\mathbb{Z}$-orientable, definably connected, definably compact, definable group, where q is some sufficiently large prime number. Then there exists a smallest type definable normal subgroup G^{00} of G of bounded index such that G/G^{00} with the logic topology is a connected, compact, Lie group. Moreover, the following hold:*

(1) *If G is abelian then G^{00} is divisible and torsion free;*

(2) $\dim G = \dim G/G^{00}$.

This theorem relating definably connected, definably compact, definable groups and connected, compact Lie groups is known as Pillay's conjecture and was stated in the paper [29], without the orientability assumption, in arbitrary o-minimal structures.

In the paper [4] by Berarducci, Otero, Peterzil and Pillay it is proved that there exists a smallest type definable normal subgroup G^{00} of G such that G/G^{00} with the logic topology is a connected, compact, Lie group. Moreover, if G is a abelian then G^{00} is divisible.

Pillay's conjecture has now been proved in three different situations: in o-minimal expansions of fields by Hrushovski, Peterzil and Pillay ([20]), in linear o-minimal expansions of ordered groups by Eleftheriou and Starchenko ([19]) and in non-linear semi-bounded o-minimal expansions of groups by Peterzil ([21]). So Pillay's conjecture holds in arbitrary o-minimal expansions of groups. In all of the three cases above the conjecture is a consequence of the following three crucial ingredients: (i) the theory of generic subsets of definably compact definable groups from [25]; (ii) the heavy model theory of definable ameanable groups from [20]; (iii) the computation of the torsion subgroups of definably connected, definably compact, abelian definable groups. Here we also have:

Theorem 1.2 *Let G be a $\mathbb{Z}/q\mathbb{Z}$-orientable, definably connected, definably compact, definable abelian group, where q is some sufficiently large prime number. Then the subgroup of m-torsion points of G is*

$$G[m] \simeq (\mathbb{Z}/m\mathbb{Z})^{\dim G}.$$

In o-minimal expansions of fields, an analogue of Theorem 1.2 without the orientability assumption was proved in the paper [16] by the first author and Otero using o-minimal singular cohomology; in the linear case it was proved in [19] by direct methods; in the non-linear semi-bounded case it was proved by a very interesting reduction to the field case in [21].

The theory of generic subsets when combined with the model theory of definable ameanable groups from [20] shows that if G is abelian then G^{00} is torsion free. Since G^{00} is also divisible, if Theorem 1.2 holds for some prime number ℓ, then

$$(G/G^{00})[\ell] = G[\ell] \simeq (\mathbb{Z}/\ell\mathbb{Z})^{\dim G}$$

and so the connected, compact, abelian Lie group G/G^{00} must be of dimension $\dim G$. Thus the conjecture holds for the abelian case. Further work of model theoretic nature combines Pillay's conjecture in the abelian case and in the definably simple case ([29]) to give the full conjecture (see [20] Remark 4 at the end of Section 8).

Note that this theory of generic subsets based on work by A. Dolich is presented in [25] only for affine definable groups but by a trick due to Peterzil and Eleftheriou (Section 8 in [21]) also works for arbitrary definably compact, definable groups in o-minimal expansions of ordered groups. Here we point out that this trick can be generalized to definably compact, definable groups in arbitrary o-minimal structures. See the remark after Proposition 3.1.

Since we also prove Theorem 1.2 for a sufficiently large prime ℓ we obtain as above Theorem 1.1. From Theorem 1.1 and the fact that G^{00} is torsion free and divisible we recover Theorem 1.2 from the special case for a sufficiently large prime ℓ.

The proof of Theorem 1.2 for a sufficiently large prime ℓ is obtained in the following way. We apply the Poincaré - Verdier duality theory from [10] to compute the o-minimal cohomology of a $\mathbb{Z}/q\mathbb{Z}$-orientable, definably connected, definably compact, abelian definable group G for a sufficiently large prime q and as a consequence we prove Theorem 1.2 for a prime ℓ such that $\ell > \dim G + 1$ and $q > \ell^{\dim G}$.

We conjecture that every definably connected, definably compact definable group is $\mathbb{Z}/q\mathbb{Z}$-orientable for every prime number q. Here, in Section 4, we prove:

Theorem 1.3 *Suppose that $\mathcal{M}$ is a non linear o-minimal expansion of an ordered group. Then every definably connected, definably compact definable group is k-orientable for every field k.*

So we obtain a different and uniform proof of Pillay's conjecture in non linear o-minimal expansions of groups.

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2 Torsions in orientable definable groups

In this section we apply the Poincaré - Verdier duality theory from [10] to compute the o-minimal cohomology of orientable, definably connected, definably compact, abelian definable groups and as a consequence we prove particular cases of Theorem 1.2 for such groups.

Let G be a definably connected, definably compact, definable group of dimension n. Then G is a definable manifold ([28]). Furthermore, by Proposition 4.4 and 4.5 in [15], G is definably normal and definably locally compact. Thus we can apply to G the orientation theory based on the o-minimal sheaf cohomology with definably compact supports introduced in [10].

So fix k a field. We say that G *has an orientation sheaf* if for every open definable subset U of G there exists a finite cover of U by open definable subsets $U_1, \dots, U_l$ of U such that for each i we have

$$H^p_c(U_i, \underline{k}) = \begin{cases} k & \text{if} \quad p = n \\ \\ 0 & \text{if} \quad p \neq n. \end{cases} \tag{1}$$

If G has an orientation sheaf, we call the sheaf $\mathcal{O}r_G$ in $\mathrm{Sh}_{\mathrm{dtop}}(G, k)$ with sections

$$\Gamma(U, \mathcal{O}r_G) = H^n_c(U, \underline{k})^\vee$$

the *orientation sheaf on G*. This a sheaf by Corollary 3.9 in [10]. Here and below, for a k-vector space N we let $N^\vee$ denote the dual k-vector space, i.e. $N^\vee = \mathrm{Hom}_k(N, k)$.

Note also that, since the o-minimal spectra $\widetilde{G}$ of G is a quasi-compact (spectral) topological space, G has an orientation sheaf if and only if for every $\beta \in \widetilde{G}$ and every open definable subset V of G such that $\beta \in \widetilde{V}$, there is an open definable subset U of V such that $\beta \in \widetilde{U}$ and

$$H^p_c(U, \underline{k}) = \begin{cases} k & \text{if} \quad p = n \\ \\ 0 & \text{if} \quad p \neq n. \end{cases}$$

If G is a definably connected, definably compact, definable group of dimension n with an orientation sheaf $\mathcal{O}r_G$, by a *k-orientation* we understand an isomorphism

$$\underline{k} \simeq \mathcal{O}r_G$$

of k-sheaves. We shall say that G is *k-orientable* if a k-orientation exists and *k-unorientable* in the opposite case.

Remark 2.1 In o-minimal expansions of fields we have o-minimal singular homology and cohomology theories satisfying the Eilenberg-Steenrod axioms adapted to the o-minimal site ([18], [30]). By [18] the o-mininal singular cohomology theory with coefficients in a field k is isomorphic to the o-minimal sheaf cohomology theory with coefficients in the constant sheaf $\underline{k}$. On the other hand the o-minimal singular homology theory can be used to obtain an orientation theory for definable manifolds ([2], [3]). (In the papers [2] and [3], orientation is defined by taking homology with coefficients in $\mathbb{Z}$ but replacing $\mathbb{Z}$ by k and considering homology groups as k-modules on gets the theory of k-orientations.)

In Subsection 3.4 in [10] we showed that in o-minimal expansions of fields the two orientation theories agree. In particular, by [2] or [16] definably connected, definably compact, definable groups in o-minimal expansions of fields are orientable.

We now observe that the existence of an orientation sheaf on G is a necessary and sufficient condition for the orientability of G:

Theorem 2.2 *Let G be a definably connected, definably compact, definable group of dimension n. Suppose that there exists a collection $\mathcal{V}$ of nonempty open definable subsets of G such that:*

(1) *for every nonempty open definable subset U of G there exists a finite cover of U by open definable subsets $V_1, \ldots, V_l \subseteq U$ in $\mathcal{V}$;*

(2) *for each $V \in \mathcal{V}$ we have*

$$H_c^p(V, \underline{k}) = \begin{cases} k & \text{if} \quad p = n \\ \\ 0 & \text{if} \quad p \neq n. \end{cases}$$

Then G is k-orientable.

Proof. It follows from the assumptions that G has an orientation sheaf $\mathcal{O}_G$. Let $V_1, \dots, V_l$ be a collection of finitely many open definable subsets in $\mathcal{V}$ which covers G. Fix $i = 1, \dots, l$. Then the restriction $\mathcal{O}_i := \mathcal{O}_{G|V_i}$ is a k-orientation sheaf for V_i. See Definition 3.10 in [10]. Since by assumption (2), $H_c^n(V_i; \underline{k}) = k$ (where $n = \dim G$), by Proposition 3.13 in [10], V_i is k-orientable, i.e. $\mathcal{O}_i \simeq \underline{k}$. See Definition 3.12 in [10].

Note that, by definition of o-minimal cohomology with definable compact supports, for every open definable subset U of V_i, we have

$$H_c^*(U, \underline{k}) = \varinjlim_{A \in c^0} H^*(U, U \setminus A, \underline{k}).$$

where c^0 is the collection of all definably connected, definable compact, definable subsets of U. Thus, by Poincaré and Alexander duality (Theorems 3.11 and 3.14 in [10]), the k-orientation $\mathcal{O}_i \simeq \underline{k}$ is uniquely determined by the induced isomorphism

$$H_c^n(V_i, V_i - v_i; \underline{k})^\vee \simeq k$$

where $v_i \in V_i$.

Without loss of generality we may assume that v_1 is the identity element of G. Fix an isomorphism $H_c^n(V_1, V_1 - v_1; \underline{k})^\vee \simeq k$ determining the k-orientation $\mathcal{O}_1 \simeq \underline{k}$. For a given i let U_i be an open definable subset of V_1 such that $v_1 \subseteq U_i$ and $v_i U_i \subseteq V_i$. Then we have an isomorphism

$$H_c^n(U_i, U_i - v_1; \underline{k})^\vee \longrightarrow H_c^n(v_i U_i, v_i U_i - v_i; \underline{k})^\vee$$

induced by the left translation by v_i. By excision axiom we get an isomorphism

$$H_c^n(V_1, V_1 - v_1; \underline{k})^\vee \simeq H_c^n(V_i, V_i - v_i; \underline{k})^\vee.$$

Now redefine the local k-orientations $\mathcal{O}_i \simeq \underline{k}$ so that they are compatible with the isomorphisms $H_c^n(V_1, V_1 - v_1; \underline{k})^\vee \simeq H_c^n(V_i, V_i - v_i; \underline{k})^\vee$. Then the local k-orientations $\mathcal{O}_i \simeq \underline{k}$ are compatible with each other and they determine, by induction on l, a global k-orientation $\mathcal{O}_G \simeq \underline{k}$ as required. $\square$

Remark 2.3 If G is a definably connected, definably compact, definable group which is defined in an o-minimal expansion of a field, then by [2], [9] or [31], there exists a collection $\mathcal{V}$ of nonempty open definable subsets of G satisfying the assumptions of Theorem 2.2.

Suppose that G is k-orientable. Then by Alexander duality in [10], if A is a definably compact, definable subset of G with l definably connected components, then

$$H^n(G, G \setminus A, \underline{k}) \simeq (k^l)^\vee.$$

So by excision, if U is an open definable subset of G such that $A \subseteq U$, then also

$$H^n(U, U \setminus A, \underline{k}) \simeq (k^l)^\vee. \tag{2}$$

We call the element $\zeta_A \in H^n(U, U \setminus A, \underline{k})^\vee$ corresponding to $(1, \ldots, 1) \in k^l$ the *fundamental class around* A. (This is well defined since the isomorphism in (2) is compatible with inclusions of open definable neighborhoods of A in G.)

Now let W be an open definable subset of G and $f : W \longrightarrow G$ a definable continuous map. Let A be a definable connected, definably compact, nonempty definable subset of G. Suppose that $f^{-1}(A)$ is a definably compact definable subset of W. Consider the image of the fundamental class around $f^{-1}(A)$ under the map

$$(f^*)^\vee : H^n(W, W \setminus f^{-1}(A), \underline{k})^\vee \longrightarrow H^n(G, G \setminus A, \underline{k})^\vee.$$

By definition of fundamental classes, we have

$$(f^*)^\vee(\zeta_{f^{-1}(A)}) = \deg_A f \zeta_A$$

for a unique element $\deg_A f \in k$ which we call the *degree of f over A*. If $W = G$, then $\deg f := \deg_G f$ is called the *degree of f*. Note that $\deg_A f = 0$ if $f^{-1}(A) = \emptyset$.

Proposition 2.4 *Let G be a definably connected, definably compact, definable group of dimension n which is k-orientable. Let W be an open definable subset of G and $f : W \longrightarrow G$ a definable continuous map. Let A be a definable connected, definably compact, nonempty definable subset of G such that $f^{-1}(A)$ is a definably compact definable subset of W. Then the following hold:*

(1) *If f is the inclusion map, then* $\deg_A f = 1$*;*

(2) *If* A_1 *is definably connected, definably compact definable subset of* G *such that* $A \subseteq A_1$ *and* $f^{-1}(A_1)$ *is a definably compact definable subset of* W*, then* $\deg_A f = \deg_{A_1} f$*;*

(3) *If* $W = W_1 \cup \cdots \cup W_l$ *where each* W_i *is an open definable subset of* W *and* $f^{-1}(A)$ *is the disjoint union of the* $f^{-1}(A) \cap W_i$*'s, then* $\deg_A f = \sum_{i=1}^{l} \deg_A f_i$*, where* $f_i = f_{|W_i} : W_i \longrightarrow G$*.*

Proof. The proof of this result is similar to that of Lemmas 4.2 (a), 4.3 and 4.5 in [16] for degree theory based on o-minimal singular homology since our fundamental classes were defined by dualizing cohomology. □

In the next result is where we need to assume that $\mathcal{M}$ is an arbitrary o-minimal structure with definable Skolem functions.

Proposition 2.5 *Let* G *be a definably connected, definably compact, abelian definable group. Suppose that there are prime numbers* ℓ *and* q *such that* $\ell > \dim G + 1$*,* $q > \ell^{\dim G}$ *and* G *is* $\mathbb{Z}/q\mathbb{Z}$*-orientable. Then the subgroup of* ℓ*-torsion points of* G *is*

$$G[\ell] \simeq (\mathbb{Z}/\ell\mathbb{Z})^{\dim G}.$$

Proof. By Theorem 1.2 in [15], there is $s \leq \dim G$ such that

$$G[\ell] \simeq (\mathbb{Z}/\ell\mathbb{Z})^{s}.$$

Consider the surjective definable homomorphism $p_\ell : G \longrightarrow G : x \mapsto \ell x$. By Theorem 3.6 in [8] this is a definable covering homomorphism and the proof of this theorem shows that there is an open definable neighborhood U of the identity element 0 of G such that $p_\ell^{-1}(U)$ is a disjoint union of open definable subsets $V_a = aU$ with $a \in \ker p_\ell$ and for each $a \in \ker p_\ell$, $p_{\ell|V_a} : V_a \longrightarrow U$ is a definable homeomorphism. Now let $V = p_\ell^{-1}(U)$ and $f = p_{\ell|V} : V \longrightarrow G$. By excision we have $\deg_0 p_\ell = \deg_0 f$ and by Proposition 2.4 (3), $\deg_0 f = \sum_{a \in \ker p_\ell} \deg_0 f_{|V_a}$. Since the composition of $f_{|V_a}$ with translation by a can be identified with the inclusion of V_a in G, it follows that $\deg_0 f_{|V_a} = 1$. Thus by Proposition 2.4 (2)

$$\deg p_\ell = \deg_0 p_\ell = |\ker p_\ell| = \ell^s.$$

By the proof of Theorem 1.2 in [15], we have

$$H^*(G; \mathbb{Z}/q\mathbb{Z}) \simeq \bigwedge[\omega_1, \ldots, \omega_t]_{\mathbb{Z}/q\mathbb{Z}}$$

for some $t \leq n = \dim G$ with $\sum_{i=1}^{t} \deg \omega_i \leq \dim G$ and the ω_i's of odd degree and primitive. Since $H^n(G, \mathbb{Z}/q\mathbb{Z}) \simeq \mathbb{Z}/q\mathbb{Z}$, we can identify the element

$\omega_G = \omega_1 \wedge \cdots \wedge \omega_t \in H^n(G, \mathbb{Z}/q\mathbb{Z})$ with the dual of the fundamental class ζ_G. As in the proof of Lemma 5.2 in [16], we see that for every $l < q$ the covering homomorphism $p_l : G \longrightarrow G : x \mapsto lx$ is such that the induced homomorphism $p_l^* : H^*(G, \mathbb{Z}/q\mathbb{Z}) \longrightarrow H^*(G, \mathbb{Z}/q\mathbb{Z})$ maps ω_G into $l^t\omega_G$. Thus

$$\deg p_\ell = \ell^t.$$

So $s = t$. But by the proof of Theorem 1.2 in [15], s is the number of the ω_i's with degree one. Therefore all the ω_i's have degree one. Since $\omega_G = \omega_1 \wedge \cdots \wedge \omega_t \in H^n(G, \mathbb{Z}/q\mathbb{Z})$ it follows that $s = t = n = \dim G$ as required. □

3 Lie groups and orientable definable groups

The goal of this section is to prove Pillay's conjecture for definably connected, definably compact, $\mathbb{Z}/q\mathbb{Z}$-orientable definable groups with q a sufficiently large prime. As explained in the introduction this follows from Proposition 2.5 and the theory of generic subsets. The following result allows us to obtain observation (1) in Section 8 of [21] showing that the theory of generic subsets of definable compact definable groups works in our case also.

Proposition 3.1 *Let G be a definably connected, definably compact, definable group of dimension n. Then the following hold:*

(1) *G has finitely many definable charts (U_i, ϕ_i)'s with each $\phi_i(U_i)$ a bounded open definable subset of M^n.*

(2) *There are finitely many pairs $U_i' \subseteq U_i$'s of open definable subsets such that $\overline{U_i'} \subseteq U_i$ and $G = \bigcup_i U_i' = \bigcup_i U_i$.*

Proof. (2) follows from the fact that G is definably normal ([15] Proposition 4.5) and the shrinking lemma ([13] Proposition 2.17).

We now prove (1). Consider the finitely many definable charts (U_i, ϕ_i)'s for G given by Pillay's construction in [28]. Then without loss of generality, U_i is a cell in G of dimension n or U_i is a translate in G of a cell in G of dimension n. In the first case, ϕ_i is the restriction of a projection from the ambient space of G onto some n coordinates. In the second case ϕ_i is the composition of a translation in G and the restriction of a projection. Thus it is enough to show that every U_i which is a cell in G is bounded.

Fix i such that U_i is a cell in $G \subseteq M^l$ and suppose that U_i is unbounded. Then there is a j such that the projection of U_i onto the j-coordinate is unbounded. Since G has definable choice ([7] Theorem 7.2) one of the following holds: (i) there is a definable map $\alpha : (e, +\infty) \subseteq M \longrightarrow G$ such that $\operatorname{im}\alpha \subseteq U_i$ and for each $t \in (e, +\infty)$, the j-coordinate $\alpha_j(t)$ of $\alpha(t)$ is t; (ii) there is a definable map $\alpha : (-\infty, d) \subseteq M \longrightarrow G$ such that $\operatorname{im}\alpha \subseteq U_i$ and for each $t \in (-\infty, d)$, the j-coordinate $\alpha_j(t)$ of $\alpha(t)$ is t. We assume (i) holds. For (ii) the proof is similar. By o-minimality we may assume that α is continuos. Since G is definably compact, the limit $\lim_{t\longrightarrow+\infty} \alpha(t)$, with respect to the topology of G, exists in G. Let a be this limit. Consider the elementary substructure $\mathcal{M}_0$ of $\mathcal{M}$ over which G (and also each of the U_i's) is defined. By replacing α by a translate $b\alpha$ of α in G where b is in the infinitesimal neighborhood over M_0 of the identity element of G, we may assume that a is a generic element of G over M_0. Now let B be a bounded open box in M^l containing a. Then $B \cap G$ is a definable neighborhood of a in G in the topology of G. Thus there is a $t_0 \in (e, +\infty)$ such that $\operatorname{im}\alpha_{|(t_0,+\infty)} \subseteq B \cap G$. But this is absurd since $\operatorname{im}\alpha_{j|(t_0,+\infty)}$ is unbounded. □

We explain how Proposition 3.1 (2) allows us to obtain observation (1) in Section 8 of [21] showing that the theory of generic subsets of definable compact definable groups works in our case also. Given any closed definable set $X \subseteq G$, each $\phi_i(X \cap \overline{U'_i})$ is closed and bounded in M^n. Using Theorem 2.1 in [25], this is sufficient to prove the result needed in that paper for the theory of generic subsets:

If $X \subseteq G$ is a closed definable subset and $\mathcal{M}_0$ is a small model then the set of $\mathcal{M}_0$-conjugates of X is finitely consistent if and only if X has a point in $\mathcal{M}_0$.

Using Proposition 3.1 instead of Lemma 7.1 in [21] allows us to obtain a generalization of a result on uniformity in parameters proved in [21] for o-minimal expansions of groups. See Lemma 7.4 in [21].

First we need the following observation. Let H be a definably, connected, definably compact definable group of dimension n. By the proof of Lemma 2.3 in [7], there is a definable chart (O, ϕ) for H with O a definable neighborhood of the identity element e_H of H and $\phi(O) = J_1 \times \cdots \times J_n$ where for each $j = 1, \ldots, n$, J_i is an open interval of the form $(-_j d_j, d_j)$ in a definable ordered divisible abelian partial group $I_j = (I_j, 0_j, +_j, -_j, <_j)$ in M. Clearly, we may assume that $\phi(e_H) = (0_1, \ldots, 0_n)$. For $\delta = (\delta_1, \ldots, \delta_n)$ with each $\delta_j >_j 0_j$, let

$$O^\delta = \{x \in O : \phi(x) = (z_1, \ldots z_n) \text{ and } \forall j\ |z_j|_j <_j \delta_j\}$$

and if D is a definable subset of H, consider the open definable neighborhood

$$O^\delta(D) = \cup\{dO^\delta : d \in D\}$$

of D in H.

Claim 3.2 *If D is a nonempty closed definable subset of H and W is an open definable neighborhood of D in H, then there exists $\delta = (\delta_1, \ldots, \delta_n)$ with each $\delta_j >_j 0_j$ such that*

$$O^\delta(D) \subseteq W.$$

Proof. This actually follows from the proof of the fact that H is definably normal ([15] Proposition 4.5) by considering the closed, disjoint definable subsets D and $H \setminus W$. □

Proposition 3.3 *Let $\{G_s : s \in S\}$ be a uniformly definable family of abelian definable groups. Then:*

(1) *The set of s for which G_s is definably connected is definable.*

(2) *The set of s for which G_s is definably compact is definable.*

Proof. The proof of (1) is similar to that of (i) in Lemma 7.4 in [21]. For (2) we may assume without loss of generality that each G_s has dimension n. By Pillay's construction in [28] there is uniformly in s, a definable family of finitely many definable charts $\{(U_{i,s}, \phi_{i,s}) : s \in S,\ i = 1, \ldots, k\}$ for the G_s's. By Proposition 3.1 we may assume that each $\phi_{i,s}(U_{i,s})$ is a bounded open definable subset of M^n.

By the proof of Lemma 2.3 in [7], there is a uniformly definable family of charts $\{(O_s, \phi_s) : s \in S\}$ for the G_s's with O_s a definable neighborhood of the identity element e_G of G_s and $\phi_s(O_s) = J_{1,s} \times \cdots \times J_{n,s}$ where for each $j = 1, \ldots, n$, $J_{i,s}$ is an open interval of the form $(-_{j,s}d_{j,s}, d_{j,s})$ in a definable ordered divisible abelian partial group $I_{j,s} = (I_{j,s}, 0_{j,s}, +_{j,s}, -_{j,s}, <_{j,s})$ in M. Clearly, we may assume that for each $s \in S$, $\phi_s(e_G) = (0_{1,s}, \ldots, 0_{n,s})$. For $s \in S$ and $\epsilon_s = (\epsilon_{1,s}, \ldots, \epsilon_{n,s})$ with each $\epsilon_{j,s} >_{j,s} 0_{j,s}$, let

$$O_s^{\epsilon_s} = \{x \in O_s : \phi_s(x) = (z_1, \ldots z_n) \text{ and } \forall j\ |z_j|_{j,s} <_{j,s} \epsilon_{j,s}\}$$

and for each $i = 1, \ldots, k$, let

$$U_{i,s}^{\epsilon_s} = U_{i,s} \setminus \cup\{u\overline{O_s^{\epsilon_s}} : u \in \overline{U_{i,s}} \setminus U_{i,s}\}.$$

Claim 3.4 *For each $s \in S$, the group G_s is definably compact if and only if there exists $\epsilon_s = (\epsilon_{1,s}, \dots \epsilon_{n,s})$ with each $\epsilon_{j,s} >_{j,s} 0_{j,s}$ such that*

$$G_s = \bigcup_{i=1}^{k} U_{i,s}^{\epsilon_s}.$$

Proof. Fix s, and assume that G_s is definably compact. For each $i = 1, \dots, k$, let $U'_{i,s}$ be given by Proposition 3.1 (2). Then $\overline{U'_{i,s}} \cap (\overline{U_{i,s}} \setminus U_{i,s}) = \emptyset$ and so $W_{i,s} = G \setminus \overline{U'_{i,s}}$ is an open definable neighborhood of $\overline{U_{i,s}} \setminus U_{i,s}$. Since by Proposition 3.1 (2), $G = \bigcup_{i=1}^{k} U'_{i,s}$ it follows from Claim 3.2, that there exists $\epsilon_s = (\epsilon_{1,s}, \dots \epsilon_{n,s})$ with each $\epsilon_{j,s} >_{j,s} 0_{j,s}$ such that $G_s = \bigcup_{i=1}^{k} U_{i,s}^{\epsilon_s}$.

For the converse, if there is an ϵ_s as above, then any (continuous) definable curve α in G_s will be eventually contained in one of the $U_{i,s}^{\epsilon_s}$. Since $\phi_{i,s}(U_{i,s}^{\epsilon})$ is bounded the (continuous) definable curve $\phi_{i,s} \circ \alpha$ has a limit $a \in M^n$, which must be in $\phi_{i,s}(U_{i,s})$. Thus $\phi_{i,s}^{-1}(a) \in G_s$ is the limit of α. □

□

4 Orientability of definably compact groups

The goal of this section is to investigate the orientability of a definably connected, definably compact definable group G defined in an o-minimal expansion of an ordered group. The best we can do so far is Theorem 1.3.

To reach our goal we require first some preliminary results which hold in arbitrary o-minimal structures.

4.1 Preliminary results

In what follows below, fix a triple (U'_i, U_i, ϕ_i) given by Proposition 3.1 and let $W' = \phi_i(U'_i)$, $W = \phi_i(U_i)$ and $\psi_W = \phi_i^{-1}$. For $0 \leq k \leq j$, let $\pi_j^k : M^{n-k} \longrightarrow M^{n-j}$ be the projection from M^{n-k} onto the first $(n-j)$ coordinates. If $k = 0$, we write π_j for π_j^0. Note that $\pi_j = \pi_j^{j-1} \circ \pi_{j-1}^{j-2} \circ \cdots \circ \pi_1^0$.

Lemma 4.1 *For $0 \leq k \leq j$, if A is a definable subset of $\pi_k \overline{W'}$, then $\pi_j^k \overline{A} = \overline{\pi_j^k A}$ is a closed and bounded, definably normal definable set. In particular, if B is a definable subset of $\overline{W'}$, then $\pi_j \overline{B} = \overline{\pi_j B}$ is a closed and bounded, definably normal definable set.*

Proof. This result is a consequence of the following claim:

Claim 4.2 *For every j, $\pi_j W'$ and $\pi_j W$ are open definable subsets of M^{n-j} such that $\pi_j W' \subseteq \pi_j \overline{W'} \subseteq \pi_j W$ and $\pi_j \overline{W'} = \overline{\pi_j W'}$ is a closed and bounded, definably normal definable set.*

Proof. We prove this by induction on j. For $j = 0$, we know from Proposition 3.1 that W' and W are open definable subsets of M^n such that $W' \subseteq \overline{W'} \subseteq W$ and $\overline{W'}$ is a closed and bounded definable set. Since $\psi_W(\overline{W'})$ is a closed definable subset of G and G is definably normal ([15] Proposition 4.5), $\overline{W'}$ is also definably normal.

For the inductive step consider $j = l+1$ and that the result holds for $j = l$. Since $\pi_{l+1} = \pi^l_{l+1} \circ \pi_l$ we have to show that if U' and U are open definable subsets of M^{n-l} such that $U' \subseteq \overline{U'} \subseteq U$ and $\overline{U'}$ is a closed and bounded, definably normal definable set, then $\pi^l_{l+1} U'$ and $\pi^l_{l+1} U$ are open definable subsets of $M^{n-(l+1)}$ such that $\pi^l_{l+1} U' \subseteq \pi^l_{l+1} \overline{U'} \subseteq \pi^l_{l+1} U$ and $\pi^l_{l+1} \overline{U'} = \overline{\pi^l_{l+1} U'}$ is a closed and bounded, definably normal definable set. Clearly under the inductive hypothesis all we have to show is that $\overline{\pi^l_{l+1} U'} \subseteq \pi^l_{l+1} \overline{U'}$ (the other inclusion is always true and the other properties follow from the equality $\pi^l_{l+1} \overline{U'} = \overline{\pi^l_{l+1} U'}$).

By the shrinking lemma ([13] Proposition 2.17) in G and using ψ_W we see that there is an open definable set W'' in M^n such that $W' \subseteq \overline{W'} \subseteq W'' \subseteq \overline{W''} \subseteq W$ and $\overline{W''}$ is definably normal. By induction hypothesis there is an open definable set U'' in M^{n-l} such that $U' \subseteq \overline{U'} \subseteq U'' \subseteq \overline{U''} \subseteq U$ and $\overline{U''}$ is definably normal. Let $x \in \overline{\pi^l_{l+1} U'}$ be such that $(\pi^l_{l+1})^{-1}(x) \cap \overline{U''} \cap \overline{U'} = \emptyset$. Since $\overline{U''}$ is definably normal, there is an open definable neighborhood V of $(\pi^l_{l+1})^{-1}(x) \cap \overline{U''}$ in $\overline{U''}$ which does not intersect $\overline{U'}$. Clearly we can assume that $V = (\pi^l_{l+1})^{-1}(V_1) \cap \overline{U''}$ where V_1 is an open definable neighborhood of x which does not intersect $\pi^l_{l+1} U'$ which is absurd. $\square$

We can now show the lemma. Clearly all we have you show is that $\overline{\pi^k_j A} \subseteq \pi^k_j \overline{A}$ (the other inclusion is always true and the other properties follow from the equality $\pi^k_j \overline{A} = \overline{\pi^k_j A}$). By Claim 4.2, $\pi_k \overline{W'} = \overline{\pi_k W'}$ is definably normal. Let $x \in \overline{\pi^k_j A}$ be such that $(\pi^k_j)^{-1}(x) \cap \pi_k \overline{W'} \cap \overline{A} = \emptyset$. Since $\pi_k \overline{W'}$ is definably normal, there is an open definable neighborhood V of $(\pi^k_j)^{-1}(x) \cap \pi_k \overline{W'}$ in $\pi_k \overline{W'}$ which does not intersect $\overline{A}$. Clearly we can assume that $V = (\pi^k_j)^{-1}(V_1) \cap \pi_k \overline{W'}$ where V_1 is an open definable neighborhood of x which does not intersect $\pi^k_j A$ which is absurd.

$\square$

We end this subsection with the following results due to Berarducci and Fornasiero ([1]) for definable sets in o-minimal expansions of groups. These can be generalized to arbitrary o-minimal structures since they rely on a result comparing sheaf cohomology and Cěch cohomology of topological spaces ([1] Fact 7.1). For a more complete treatment of Cěch cohomology in o-minimal structures we refer the reader to [17], but Appendix D in [1] is also useful.

Lemma 4.3 ([1]) *Let X be a definable set. If there exists a finite covering $\mathcal{U} = \{U_i : i \in I\}$ of X by open definable subsets with $H^p(U_J; \underline{k}) = 0$ for every $p > 0$ and $H^0(U_J; \underline{k}) = k$ for every finite $J \subset I$ with $U_J := \bigcap_{j \in J} U_j$ non-empty, then $H^i(X; \underline{k})$ is isomorphic to the i-th simplicial cohomology group of the nerve $N(\mathcal{U})$ of the covering with coefficients in k.*

Moreover they proved in [1] also the following result:

Lemma 4.4 ([1]) *Let $X \subset Y$ be definable sets. Suppose that there are finite covering $\mathcal{U} = \{U_i : i \in I\}$ of X and $\mathcal{V} = \{V_i : i \in I\}$ of Y by open definable subsets indexed by the same finite set I such that:*

(1) *$U_i \subset V_i$ for all $i, j \in I$.*

(2) *For all finite $J \subset I$, $U_J := \bigcap_{j \in J} U_j$ is non empty (i.e. the natural map among the nerves of the covering is an isomorphism).*

(3) *For each finite $J \subset I$ the sets U_J and $V_J := \bigcap_{j \in J} V_j$ are either empty or connected, and for all $q > 0$, $H^q(U_J; \underline{k}) = H^q(V_J; \underline{k}) = 0$.*

Then the inclusion map $X \subset Y$ induces an isomorphism $H^*(Y; \underline{k}) \to H^*(X; \underline{k})$.

4.2 Proof of Theorem 1.3

Here we assume that $\mathcal{M}$ is an o-minimal expansion of an ordered group $(M, 0, +, <)$. In this setting it is proved in [1] that, given a closed and bounded definable set X the cohomology $H^i(X; \underline{k})$ is finitely generated for every i. In the proof of this result there is a construction which will be very useful here also. Thus we include here some of the results and their proofs from [1] relativized to definable subsets of $\pi_j W'$ from Subsection 4.1 above.

Lemma 4.5 ([1]) *Every cell C in $\pi_j W'$ is acyclic, i.e. $H^p(C; \underline{k}) = 0$ for $p > 0$ and $H^0(C; \underline{k}) = k$.*

This lemma follows from: (i) the homotopy axiom for o-minimal cohomology ([13]); (ii) the fact that for every cell is there exists a definable deformation retract onto a cell of strictly lower dimension ([1] Lemma 5.2) and (iii) an interval is definably contractible to a point ([1] Lemma 5.1 and the remark after Corollary 5.3 in [1]).

Lemma 4.6 ([1]) *Let $C \subseteq \pi_j W'$ be a (bounded) cell of dimension m. There is a definable family $\{C_t : t > 0\}$ of definably compact sets $C_t \subset C$ such that:*

(1) $C = \bigcup_{t>0} C_t$.

(2) *If $0 < t' < t$ then $C_t \subset C_{t'}$ and the inclusion induces an isomorphism $H^p(C\backslash C_t; \underline{k}) \to H^p(C\backslash C_{t'}; \underline{k})$.*

(3) *For every $t > 0$, $C\backslash C_t$ has the same cohomology groups of an $m-1$ dimensional sphere, namely*

$$H^p(C\backslash C_t; \underline{k}) = \begin{cases} k & \text{if} \quad p \in \{0, m-1\} \\ \\ 0 & \text{if} \quad p \notin \{0, m-1\}. \end{cases}$$

Proof. We define the definably family $\{C_t : t > 0\}$ by induction on the dimension of the cells in the following way.

(1) If $l := n - j = 1$ and C is a singleton in $\pi_j W'$ we define $C_t = C$.

(2) If $l := n - j = 1$ and $C = (a, b)$, then $C_t = [a + \gamma_t, b - \gamma_t]$ where $\gamma_t = \min\{\frac{a+b}{2}, t\}$, (in this way C_t is non empty).

(3) Let $l := n - j > 1$ and $C = \Gamma(f)$, where $f : B \to M$ is a continuous definable map and $B \subseteq \pi_{j+1} W'$ is a bounded cell. By induction B_t is defined so we define $C_t = \Gamma(f_{|B_t})$ (the graph of the restriction of the f on B_t.)

(4) Let $l := n - j > 1$ and $C = (f, g)_B$, where $f, g : B \to M$ are continuous definable maps on $B \subseteq \pi_{j+1} W'$ a bounded cell such that $f < g$. By induction B_t is defined. We put $C_t = [f + \gamma_t, g - \gamma_t]_{B_t}$, where $\gamma_t := \min(\frac{f-g}{2}, t)$.

We observe that from this construction we obtain:

Claim 4.7 *For each $t > 0$ there is a covering $\mathcal{U}_C = \{U_i : i \in I\}$ of $C\backslash C_t$ such that:*

(1) *The index set I is the family of the closed faces of an m-dimensional cube, where $m = \dim(C)$. (So $|I| = 2m$).*

(2) *If $F \subset I$, then $U_F := \bigcap_{i \in F} U_i$ is either empty or a cell. (So in particular $H^p(U_F; \underline{k}) = 0$ for $p > 0$ and, if $U_F \neq \emptyset, H^0(U_F; \underline{k}) = k$.)*

(3) *For* $F \subset I$, $U_F \neq \emptyset$ *iff the faces of the cubes belonging to* F *have a non-empty intersections. (So the nerve of* $\mathcal{U}$ *is isomorphic to the nerve of a covering of an* m*-cube by its closed faces.)*

Proof. To prove that there is a covering satisfying the properties above, we define $\mathcal{U}_C$ by induction on the dimension $n-j$ of $\pi_j W'$. We distinguish four cases according to definition of the C_t.

(1) If $l := n-j = 1$ and C is a singleton, then $\mathcal{U}_\mathcal{C}$ is the covering consisting of one open set (given by the whole space C).

(2) If $l := n-j = 1$ and $C = (a, b)$, then $C \backslash C_t$ is the union of the two open subsets $(a, a+\gamma_t)$ and $(b-\gamma_t, b)$, and we define $\mathcal{U}_C$ as the covering consisting of these two sets.

(3) Let $l := n-j > 1$ and $C = \Gamma(f)$, where $f : B \to M$. By induction we have a covering $\mathcal{V}$ of $B \backslash B_t$ with the stated properties, and we define $\mathcal{U}_C$ to be a covering of $C \backslash C_t$ induced by the natural homeomorphism between the graph of f and its domain.

(4) Let $l := n-j > 1$ and $C = (f, g)_B$. By definition $C_t = (f+\gamma_t, g-\gamma_t)_{B_t}$. By induction we have that $B \backslash B_t$ has a covering $\mathcal{V} = \{V_j : j \in J\}$ with the stated properties, where J is the set of closed faces of the cube $[0,1]^{m-1}$. Define a covering $\mathcal{U}_C = \{U_i : i \in I\}$ of $C \backslash C_t$ as follows. As index set I we take the closed faces of the cube $[0,1]^m$. Thus $|I| = |J|+2$, the two extra faces of the corresponding to the"top" and "bottom" face of $[0,1]^m$. We associate to the top face the open set $(g-\gamma_t, g)_B \subset C \backslash C_t$ and the bottom face the open set $(f, f+\gamma_t)_B \subset C \backslash C_t$. The other open sets of the covering are the preimages of the sets V_j under the projection $M^{n-j} \to M^{n-j-1}$. This defines a covering of $C \backslash C_t$ with the stated properties.

□

Properties (2) and (3) of the lemma now follow from Claim 4.7 and Lemmas 4.3, 4.4 and 4.5.

□

Lemma 4.8 *Let $C \subseteq \pi_j W'$ be a (bounded) cell of dimension m. Then*

$$H^p_c(C;\underline{k}) = \begin{cases} k & \text{if} \quad p = m \\ \\ 0 & \text{if} \quad p \neq m. \end{cases}$$

Proof. Let c be the family of definably compact, definable sets. Then by definition of o-minimal sheaf cohomology with definably compact supports,

$$H^*_c(C,\underline{k}) = \varinjlim_{U\subseteq C,\ C\setminus U\in c} H^*(C,U,\underline{k}).$$

Since for every $U \subseteq C$ such that $C\setminus U \in c$ there is $t > 0$ such that $U \subseteq C\setminus C_t$ (as each $C_t \in c$) , we have

$$H^*_c(C,\underline{k}) = \varinjlim_{t>0} H^*(C, C\setminus C_t, \underline{k}).$$

Now the result follows from the exactness axiom

$$\ldots \longrightarrow H^l(C,C\setminus C_t;\underline{k}) \longrightarrow H^l(C;\underline{k}) \longrightarrow H^l(C\setminus C_t;\underline{k}) \longrightarrow H^{l+1}(C,C\setminus C_t;\underline{k}) \longrightarrow \ldots$$

and Lemmas 4.5 and 4.6. (Note that in [13] the exactness axiom is stated for closed definable subsets but it holds for arbitrary definable subsets with the same proof). □

Proof of Theorem 1.3. Let G be a definably connected, definably compact, definable group of dimension n. Let $\mathcal{V}$ be the collection of all nonempty open definable subsets of G such that $V \in \mathcal{V}$ if and only if there exists a U_i' given by Proposition 3.1 (2) such that $V \subseteq U_i'$ and $\phi_i(V)$ is an open cell of a cell decomposition of $\phi_i(U_i')$. By Lemma 4.8 and the following claim, assumptions (1) and (2) of Theorem 2.2 hold as required.

Claim 4.9 ([11]) *Suppose that $\mathcal{M}$ is a non linear o-minimal expansion of an ordered group. Then every nonempty, bounded, open definable subset U of M^n can be covered by finitely many open cells in M^n contained in U.*

This is a generalization to non linear o-minimal expansions of ordered groups of Wilkie's result in o-minimal expansions of fields in [31].

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