On torsion points of locally definable groups in o-minimal structures

Mário J. Edmundo *

CMAF Universidade de Lisboa

Av. Prof. Gama Pinto 2

1649-003 Lisboa, Portugal

October 25, 2003

Abstract

In this paper we study the structure of m-torsion points of connected locally definable abelian groups in o-minimal expansions of fields.

*Supported by the FCT grant SFRH/BPD/6015/2001 while a post-doctoral research fellow at the Mathematical Institute, University of Oxford. *MSC*: 03C64; 20E99. *Keywords and phrases:* O-minimal structures and locally definable groups. Revised February 11, 2005.

1 Introduction

Throughout this paper, $\mathcal{N}$ will be an $\aleph_1$-saturated o-minimal structure and definable will mean $\mathcal{N}$-definable (possibly with parameters). We are interested here in understanding the subgroup of m-torsion points of a connected locally definable group. A locally definable group is roughly a group whose underlying set is a union of definable sets and the graphs of the group operations are unions of definable sets. Locally definable groups are a special case of what is called in the literature $\bigvee$-definable groups and they occour quite often in connection with the study of definable groups (see [e2], [ps], [pst] and [pps]).

By [e3], many results from the theory of definable groups, which includes real algebraic groups and semi-algebraic groups, have an analogue in the theory of locally definable groups. Among these we have the following properties:

(TOP) every locally definable group G over A has a unique locally definable topological structure over A such that the group operations are continuous and the locally definable homomorphisms are also continuous;

(DCC) the descending chain condition for compatible locally definable subgroups over A of a locally definable group G over A;

(QT) existence in the category of locally definable groups over A of the quotient of a locally definable group over A by a compatible locally definable normal subgroup over A together with the existence of a corresponding locally definable section over A;

(AB) every locally definable group G over A of positive dimension has a compatible locally definable abelian subgroup over A of positive dimension.

The definable analogues of (TOP), (DCC) and (AB) were proved in [p]. (TOP) for locally definable groups over A is essentially the same as (TOP) for $\bigvee$-definable group over A which was proved in [pst]. Property (QT) for definable groups is from [e1]. In Section 2 here, we define locally definable groups and present some of the properties of these groups that we will need. We then assume that $\mathcal{N}$ is an o-minimal expansion of a field and prove the following results:

Theorem 1.1 *Let G be a connected locally definable abelian group over A and let m be a natural number with prime decomposition $p_1^{k_1} \cdot \dots \cdot p_l^{k_l}$ such that $p_1, \dots, p_l \geq \dim G + 2$. Then the subgroup $G[m]$ of m-torsion points of G is a finite locally definable subgroup over A with at most $2l \dim G$ generators.*

Roughly a locally definable group G is connected if it has no proper locally definable subgroup H such that $(G : H) < \aleph_1$. See Section 2 for details on this notion. Since by [e3] Corollary 3.16 (i), every torsion point of a connected locally definable group is in the centre of the group, Theorem 1.1 implies the following:

Corollary 1.2 *Let G be a connected locally definable group over A and let m be a natural number with prime decomposition $p_1^{k_1} \cdot \dots \cdot p_l^{k_l}$ such that $p_1, \dots, p_l \geq \dim G + 2$. Then the subgroup $G[m]$ of m-torsion points of G is a finite definable subgroup over A.*

Corollary 1.2 is a special case of the following property:

(TOR) If G is a connected locally definable group over A, then for all $m \in \mathbb{N}$, the subgroup $G[m]$ of m-torsion points of G is a finite definable subgroup over A.

For definable groups property (TOR) is proved in [s] using the o-minimal Euler characteristic for definable sets defined by van den Dries [vdd] using the cell decomposition theorem. This is an Euler-Grothendieck characteristic for the definable category in the sense of [ks]. In our more general context, there is no generalization of the o-minimal Euler characteristic since the Grothendieck ring of the category of locally definable sets with locally definable maps is trivial.

As we saw in [eo] the structure of the subgroup $G[m]$ of m-torsion points of a definable group G is closely related to the o-minimal singular cohomology groups $H^*(G;\mathbb{Q})$ of G over $\mathbb{Q}$. The property we require for $H^*(G;\mathbb{Q})$ is that it a Hopf $\mathbb{Q}$-algebra and this is proved using the fact that the o-minimal singular homology groups $H_*(G)$ are bounded and of finite type i.e., $H_i(G)$ is a finitely generated abelian group and $H_i(G) = 0$ for all $k > \dim G$. More precisely, we require that $H_*(G)$ is of finite type since the multiplication on $H^*(G;\mathbb{Q})$ is given by the cup product and the comultiplication is given by $\mu = (\alpha')^{-1} \circ u^*$ where $u : G \times G \longrightarrow G$ is the multiplication map on G and α' is the cross product homomorphism in the Künneth formula for the o-minimal singular cohomology whose validity requires the groups $H_*(G)$ to be of finite type. (Compare with the classical case in [d] Chapter VI, Proposition 12.16).

It is easy to generalize Woerheide's ([Wo]) construction of the o-minimal singular homology and cohomology in the definable category to the category of locally definable spaces (o-minimal analogues of locally semi-algebraic spaces from [dk]). Similarly the (co)homology products from [eo] can also be generalized. However, for a locally definable group G over A, which by (TOP) is a locally definable space, we cannot guarantee that $H_*(G)$ is of finite type. To overcame this difficulty, we will dualize and consider instead the o-minimal singular homology groups $H_*(G; \mathbb{Z}/p\mathbb{Z})$. When G is abelian, $H_*(G; \mathbb{Z}/p\mathbb{Z})$ can be made into a Hopf $\mathbb{Z}/p\mathbb{Z}$-algebra with multiplication $u_* \circ \alpha''$ and comultiplication $(\alpha'')^{-1} \circ \Delta_G$ where $\Delta_G : G \longrightarrow G \times G$ is the diagonal and α'' is the cross product homomorphism in the Künneth formula for the o-minimal singular homology whose validity does not require the groups $H_*(G)$ to be of finite type. (Compare with the classical case in [d] Chapter VI, Corollary 12.10). In Section 3, we recall the basic theory of Hopf algebras and show how to attach a bounded Hopf algebra to every connected locally definable abelian group.

In Section 4, we use the theory of locally definable covering homomorphisms from [e2] together with a modified version of the Hopf-Leray-Borel theorem classifying Hopf algebras of finite type over perfect fields, to prove Theorem 1.1 by considering the Hopf algebras $H_*(G; \mathbb{Z}/p\mathbb{Z})$ with $p > \dim G + 1$. The reason we consider the coefficient ring $\mathbb{Z}/p\mathbb{Z}$ instead of $\mathbb{Q}$ is due to the fact that, unlike in the definable case in [eo], we cannot guarantee a prior that the o-minimal fundamental group $\pi_1(G)$ of the connected locally definable abelian group G over A is a torsion-free finitely generated abelian group. In the definable case ones uses the fact that a definably connected definable abelian group G is divisible to conclude that $\pi_1(G)$ is torsion-free!

The fact that every connected definable abelian group G is divisible, is known as property (DIV), and is a consequence of (TOR): since $G[m]$ is a finite definable subgroup, mG is a definable subgroup of G of dimension $\dim G$. By [p], $(G : mG) < \aleph_0$. But since G is definably connected, again by [p], we must have $mG = G$ as required. This argument fails in the case of locally definable groups, because a locally definable group G can have a locally definable subgroup H such that $\dim H = \dim G$ and $(G : H)$ is not bounded by $\aleph_1$. So the following question remains open.

Question: Is every connected locally definable abelian group divisible?

In the process of proving Theorem 1.1 we actually show the following result (see Lemma 4.1).

Theorem 1.3 *Let G be a connected locally definable abelian group over A. Then the following hold.*

(1) If m is a natural number with prime decomposition $p_1^{k_1} \cdot \dots \cdot p_l^{k_l}$ such that $p_1, \dots, p_l \geq \dim G + 2$, then the subgroup $\pi_1(G)[m]$ of m-torsion points of $\pi_1(G)$ is finite with at most $l \dim G$ generators.

(2) The subgroup $\pi_1(G)[\infty]$ of torsion free points of $\pi_1(G)$ is finitely generated with at most $\dim G$ generators.

If we assume (DIV), then combining Theorem 1.3 with [e2] Theorem 1.1 we get the following result.

Theorem 1.4 *Let G be a divisible, connected locally definable abelian group over A. Then the following hold:*

(i) the o-minimal fundamental $\pi_1(G)$ of G is isomorphic to $\mathbb{Z}^s$ where $s \leq \dim G$ and;

(ii) for all $m \in \mathbb{N}$, the subgroup $G[m]$ of m-torsion points of G is a finite definable subgroup over A isomorphic to $(\mathbb{Z}/m\mathbb{Z})^s$.

Finally, in the concluding remarks in Section 5, we prove the following o-minimal homology analogue of the structure theorem for definably compact abelian groups from [eo].

Theorem 1.5 *Let G be a definably compact, definably connected, definable abelian group of dimension n. Then the following hold:*

(i) for all $m \in \mathbb{N}$, the subgroup $G[m]$ of m-torsion points of G is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^n$;

(ii) the o-minimal fundamental $\pi_1(G)$ of G is isomorphic to $\mathbb{Z}^n$;

(iii) if K_p is a perfect field of characteristic p (a prime or zero), then $H_*(G; K_p) = \bigwedge[y_{1:1}, \dots, y_{n:1}]_{K_p}$ with $\deg y_{l:1} = 1$ for all $l = 1, \dots, n$.

Note that the proof we present here of Theorem 1.5, does not use o-minimal cohomology, and so, this is a new proof of (i) and (ii).

2 Locally definable groups

In this section we define locally definable groups and present some of the properties of these groups that we will need in the paper.

Recall that, by [pst], a group $G = (G, \cdot)$ is a $\bigvee$-*definable group* over $A \subseteq N$, where $|A| < \aleph_1$, if there is a collection $\{Z_i : i \in I\}$ of definable subsets of N^n, all definable over A such that: (i) $G = \cup\{Z_i : i \in I\}$; (ii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$ and (iii) the restriction of the group multiplication to $Z_i \times Z_j$ is a definable map into N^n. We modify this definition slightly in the following way.

Definition 2.1 A group $(G, \cdot)$ is a *locally definable group over* A, with $A \subseteq N$ and $|A| < \aleph_1$, if there is a collection $\{Z_i : i \in I\}$ of definable subsets of N^n, all definable over A such that: (i) $G = \cup\{Z_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $G = \cup\{Z_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$ and (iv) the restriction of the group multiplication to $Z_i \times Z_j$ is a definable map into N^n.

A homomorphism $\alpha : G \longrightarrow H$ between locally definable groups over A is called a *locally definable homomorphism over* A if for every definable subset $Z \subseteq G$ defined over A, the restriction $\alpha_{|Z}$ is a definable map over A.

As in the $\bigvee$-definable case, the locally definable group G over A does not depend on the choice of the collection $\{Z_i : i \in I\}$ in the sense that if $G = \cup\{Y_j : j \in J\}$ with each Y_j definable over B, $|B| < \aleph_1$, then by saturation and the definition the following hold: (i) every Y_j is contained in some Z_i and vice-versa and (ii) there is $J_0 \subseteq J$ with $|J_0| < \aleph_1$ and $G = \cup\{Y_j : j \in J_0\}$. *For this reason we will always assume from now on that* $|I| < \aleph_1$. This condition is used in [e3] to prove property (QT).

Also note that if $\alpha : H \longrightarrow G$ is a locally definable homomorphism over A between locally definable groups over A and if K is a locally definable subgroup of G over A. Then $\alpha(H)$ is a locally definable group over A and $\alpha^{-1}(K)$ is a locally definable subgroup of H over A.

Before we proceed any further we recall the two main examples of locally definable groups.

Example 2.2 The following are the two main examples of locally definable groups over A, with $A \subseteq N$ and $|A| < \aleph_1$.

(1) The locally definable groups over A of dimension zero: Let $\{Z_i : i \in I\}$ be a collection of finite subsets of N^k all of which defined over A such that for all $i, j \in I$ there is $k \in I$ with $Z_i \cup Z_j \subseteq Z_k$ and $(\mathcal{Z}, \cdot)$ is an abstract group, where $\mathcal{Z} = \cup\{Z_i : i \in I\}$, and there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Z_i : i \in I_0\}$. Then $(\mathcal{Z}, \cdot)$ is a locally definable group over A of dimension zero.

(2) The locally definable groups over A which are the subgroups of (type) definable groups: Let $(G, \cdot)$ be a (type) definable group over $B \subseteq A$; let $\{Z_i : i \in I\}$ be a collection of definable subsets of G all of which defined over A such that for all $i, j \in I$ there is $k \in I$ with $Z_i \cup Z_j \subseteq Z_k$, $(\mathcal{Z}, \cdot)$ is a subgroup of $(G, \cdot)$, where $\mathcal{Z} = \cup\{Z_i : i \in I\}$, and there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Z_i : i \in I_0\}$. Then $(\mathcal{Z}, \cdot)$ is a locally definable group over A.

The proof of [pst] Proposition 2.2 also show the following theorem.

Theorem 2.3 *Let $G \subseteq N^k$ be a locally definable group over A. Then there is a uniformly definable family $\{V_s : s \in S\}$ of definable subsets of G defined over A and containing the identity element of G and there is a unique topology τ on G such that: (i) $\{V_s : s \in S\}$ is a basis for the τ-open neighbourhoods of the identity element of G; (ii) (G, τ) is a topological group and (iii) every generic element of G has an open definable neighbourhood $U \subseteq N^k$ such that $U \cap G$ is τ-open and the topology which $U \cap G$ inherits from τ agrees with the topology it inherits from N^k.*

In Theorem 2.3, by a uniformly definable family $\{V_s : s \in S\}$ of definable subsets of G defined over A we mean that S is definable over A and there is a definable subset of $N^k \times S$ over A such that the fiber over s is V_s for each $s \in S$.

If $G = \cup\{Z_i : i \in I\}$ is a locally definable group over A, we define the *dimension* of G by $\dim G = \max\{\dim Z_i : i \in I\}$. As in [pst] Lemma 2.6 we see that the following result holds.

Theorem 2.4 *Let G be a locally definable group over A and H a locally definable subgroup of G over A. Then the following holds: (i) the τ-topology* *on H is the subspace topology induced by the τ-topology on G; (ii) H is closed in G in the τ-topology and (iii) H is open in G in the τ-topology if and only if* $\dim H = \dim G$.

The proof of [pst] Lemma 2.8 gives the following theorem.

Theorem 2.5 *Any locally definable homomorphism between locally definable groups is a continuous locally definable homomorphism with respect to the τ-topology.*

Theorems 2.3, 2.4 and 2.5 are called property (TOP) for locally definable groups since they generalise the corresponding property for definable groups. *From now on, whenever we use topological notions on a locally definable group we are referring to the τ-topology.*

We will now introduce the notion of compatible locally definable subsets of a locally definable group. This notion is useful for understanding connectedness in the context of locally definable groups.

Definition 2.6 A set Z is a *locally definable set over A* where $A \subseteq N$ and $|A| < \aleph_1$ if there is a collection $\{Z_i : i \in I\}$ of definable subsets of N^n, all definable over A such that: (i) $Z = \cup\{Z_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $Z = \cup\{Z_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$.

A map $\alpha : Z \longrightarrow X$ between locally definable sets over A is called a *locally definable map over A* if for every definable subset $V \subseteq Z$ defined over A, the restriction $\alpha_{|V}$ is a definable map over A.

By saturation, the set Z does not depend on the choice of the collection $\{Z_i : i \in I\}$ in the sense that if $Z = \cup\{Y_j : j \in J\}$ with each Y_j definable over B, $|B| < \aleph_1$, then the following hold: (i) every Y_j is contained in some Z_i and vice-versa and (ii) there is $J_0 \subseteq J$ with $|J_0| < \aleph_1$ and $Z = \cup\{Y_j : j \in J_0\}$. *For this reason we will always assume from now on that $|I| < \aleph_1$.*

Let $\alpha : Z \longrightarrow X$ be a locally definable map over A between locally definable sets over A and let Y be a locally definable subset of X over A. Then $\alpha(Z)$ is a locally definable set over A and $\alpha^{-1}(Y)$ is a locally definable subset of Z over A. If $Z = \cup\{Z_i : i \in I\}$ is a locally definable set over A, we define the *dimension* of Z by $\dim Z = \max\{\dim Z_i : i \in I\}$.

Definition 2.7 Let G be a locally definable group over A and let H be a locally definable subgroup (resp., subset) of G over A. We say that H is a *compatible locally definable subgroup (resp., subset)* if for every open definable subset U of G over A, the set $H \cap U$ is a definable subset of G over A.

For example, if H is a definable subgroup (resp., subset) of G over A, then H is a compatible locally definable subgroup (resp., subset) of G.

We now turn to the notion connectedness. The following definition is the analogue of [pst] Definition 2.12.

Definition 2.8 Let G be a locally definable group over A. We say that a set $Z \subseteq G$ is *connected* if there is no definable subset $U \subseteq G$ over A such that $U \cap Z$ is a nonempty proper subset of Z which is closed and open in the topology induced on Z by G.

The next remark can be proved in exactly the same way as [pst] Lemmas 2.13 and 2.14.

Remark 2.9 Let G be a locally definable group over A. Then the following hold:

(1) Every definable open subset $Z \subseteq G$ over A can be partitioned into finitely many connected definable subsets of G over A.

(2) There is a locally definable subgroup G' of G over A which is connected and such that $\dim G' = \dim G$.

As pointed out in [pst], the connected locally definable subgroups given by Remark 2.9 (2) are not unique. In fact, let $\mathcal{N}$ be a non standard model of the theory of the ordered additive group of real numbers, $G = (N^2, +)$, $G' = \{(x, y) \in N^2 : \text{there exists } n \in \mathbb{N} \text{ such that } -n < x < n\}$ and $G'' = \{(x, y) \in N^2 : \text{there exists } n \in \mathbb{N} \text{ such that } -n < y < n\}$. Then G' and G'' are two distinct connected locally definable subgroups of G over $\mathbb{Z}$.

Nevertheless, we have the following generalization of [pst] Lemma 2.15 (iii). See [e2] Proposition 2.18.

Proposition 2.10 *Let G be a locally definable group over A. Then there is a unique connected compatible locally definable normal subgroup G^0 of G over A with dimension* $\dim G$*. Moreover, the following hold:*

(i) G^0 contains all connected locally definable subgroups of G over A;

(ii) G^0 is the smallest compatible locally definable subgroup of G over A such that $(G : G^0) < \aleph_1$

(iii) there is a locally definable subset $\{x_s : s \in S\}$ of G over A such that $G = \cup\{x_s G^0 : s \in S\}$ (disjoint union).

3 O-minimal singular homology

From now on we will assume that $\mathcal{N}$ is an o-minimal expansion of a field.

3.1 O-minimal singular homology groups

In o-minimal expansions of real closed fields, Woerheide ([Wo]) constructs the o-minimal singular homology (H_*, d_*) with coefficients in $\mathbb{Z}$ satisfying the o-minimal Eilenberg-Steenrod homology axioms (the analogues of the Eilenberg-Steenrod axioms for the category of definable sets with definable continuous maps).

The definition of the o-minimal homology groups is quite easy, but the verification of the axioms is very difficult as we now explain. Indeed, the construction is essentially the same as for the standard singular homology, only with the word "definable" added here and there. But the standard proof of the excision axiom, based on the repeated barycentric subdivisions and the Lebesgue number property for the standard simpleces Δ^n, fails and the difficulty is avoided by the use of the o-minimal triangulation theorem and his construction of the o-minimal simplicial homology.

Here we will point out how to generalise Woerheide's construction and define the o-minimal singular homology groups $H_*(G)$ of a locally definable group G over A. Observe that one could in fact define o-minimal singular homology (H_*, d_*) with coefficients in $\mathbb{Z}$ in the category of locally definable spaces over A satisfying the corresponding o-minimal Eilenberg-Steenrod homology axioms. However this level of generality will not be necessary here.

So let $G \subseteq N^k$ be a locally definable group over A. If $h : X \longrightarrow G$ is a map such that as a map into N^k is definable, we say that h is a definable map. We say that h is a continuous definable map, if it is continuous when we take the τ-topology on G.

For each $m \geq 0$, consider the abelian group $S_m(G)$ freely generated by the singular (continuous) definable simplixes $\sigma : \Delta^m \longrightarrow G$, where $\Delta^m = \{(t_0, \ldots, t_m) \in N^{m+1} : \sum_i t_i = 1, t_i \geq 0\}$ is the standard m-dimensional simplex. The boundary operator $\partial_{m+1} : S_{m+1}(G) \longrightarrow S_m(G)$ (morphism of degree -1) is defined as in the classical case making $S_*(G)$ a free chain complex. Also, a locally definable continuous map $f : G \longrightarrow K$ over A between locally definable groups over A induces a chain map $f_\sharp : S_*(G) \longrightarrow S_*(K)$ (i.e., a morphism of degree zero satisfying $f_\sharp \circ \partial_* = \partial_* \circ f_\sharp$). The graded group $H_*(G)$ is defined as the homology of the chain complex $S_*(G)$. A locally definable continuous map $f : G \longrightarrow K$ over A between locally definable groups over A induces a homomorphism $f_* : H_*(G) \longrightarrow H_*(K)$ of graded groups (via $f_\sharp$).

This construction easily gives, as in the classical case treated in [d] Chapter VI, Section 7, the o-minimal singular homology with coefficients in any $\mathbb{Z}$-module L. Indeed, if $f : G \longrightarrow K$ is a locally definable continuous map over A between locally definable groups over A, one defines the *o-minimal singular homology with coefficients in* L by

$$H_m(G; L) = H_m(S_*(G) \otimes L)$$

and $f_* : H_m(G; L) \longrightarrow H_m(K; L)$ is the homomorphism induced by $f_\sharp \otimes \mathrm{id}$.

The next result follows from the homological algebra universal coefficients theorem in [d] Chapter VI, Theorem 4.2. For details compare with [d] Chapter VI, Section 7.

Proposition 3.1 (Universal Coefficients Theorem) *Suppose that G is a locally definable group over A and L is a $\mathbb{Z}$-module. Then, for every $n \in \mathbb{Z}$, there are natural exact sequences*

$$0 \to H_n(G) \otimes L \xrightarrow{\alpha} H_n(G; L) \to \mathrm{Tor}(H_{n-1}(G), L) \to 0.$$

By construction of the o-minimal singular homology groups $H_*(G)$ of a locally definable group G over A, one can also develop the theory of products for the o-minimal singular homology in the same way as in the classical case treated in [d] Chapter VI and VII. Here we recall only the o-minimal singular homology cross product since this will be used in the proof of our main result.

Theorem 3.2 (Künneth Formula for Homology) *Assume that G and K are locally definable groups over A and L is a $\mathbb{Z}$-module. Then, for all $n \in \mathbb{Z}$, there is an isomorphism*

$$\alpha'' : \sum_{i+j=n} H_i(G; L) \otimes H_j(K; L) \longrightarrow H_n(G \times K; L).$$

Proof. As in the definable case treated in [eo] Proposition 3.2, we have the o-minimal version of the Eilenberg-Zilber theorem: there are unique (up to chain homotopy) natural chain equivalences $S_*(G\times K) \longrightarrow S_*(G)\otimes S_*(K)$ and $S_*(G)\otimes S_*(K) \longrightarrow S_*(G\times K)$ inverse to each other. Any such chain map is called an EZ map. Furthermore, the EZ maps are commutative, associative and preserve units as in [d] Chapter VI, 12.1 - 12.5. The isomorphism α'' is defined using the EZ maps as its classical analogue in [d] Chapter VI, Corollary 12.12 in the same purely algebraic way. □

The homomorphism α'' from Theorem 3.2 is called the *homology (external) cross product* and $\alpha''(a \otimes b)$ is denoted $a \times b$.

Theorem 3.3 *The homology cross product satisfies the following properties:*

(1) **Naturality**. $(f \times g)_*(\alpha \times \beta) = (f_*\alpha) \times (g_*\beta)$

(2) **Skew-commutativity**. $t_*(\alpha \times \beta) = (-1)^{\deg\alpha \deg\beta}\beta \times \alpha$ *where* $t : G \times K \longrightarrow K \times G$ *commutes factors.*

(3) **Associativity**. $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$.

(4) **Units**. $1 \times \alpha = \alpha \times 1 = \alpha$.

Proof. The proof of this result is purely algebraic and similar to that of its classical analogue in [d] Chapter VII, Section 2. In fact, these properties are consequences of the corresponding properties for the EZ maps. □

3.2 $H_k(G)$ for $k = 0, 1$ and $k > \dim G$

Let G be a locally definable group over A. Here we will compute $H_k(G)$ for $k = 0, 1$ and $k > \dim G$.

Recall from [e2] that a *definable path* in G is a continuous definable map $\alpha : [0,1] \longrightarrow G$; we say that G is *definably path connected* if for every $u, v \in G$ there is a definable path $\alpha : [0,1] \longrightarrow G$ such that $\alpha(0) = u$ and $\alpha(1) = v$.

Proposition 3.4 *If G is a connected locally definable group over A, then $H_0(G) \simeq \mathbb{Z}$.*

Proof. By [e2] Lemma 4.5, G is definably path connected. Now the result can be obtained in exactly the same way as its classical analogue in [d] Chapter III, Proposition 4.11. □

A definable path $\alpha : [0,1] \longrightarrow G$ is called a *definable loop at e_G* if $\alpha(0) = e_G = \alpha(1)$. Two definable paths α and β in X are said to be *definably homotopic* if there is a definable map $H : [0,1] \times [0,1] \longrightarrow G$ such that $H(0,t) = \alpha(t)$ and $H(t,1) = \beta(t)$ for all $t \in [0,1]$. The equivalence class of α under this equivalence relation if denoted by $[\alpha]$.

The set of equivalence classes under definable homotopies of definable loops at e_G is the o-minimal fundamental group $\pi_1(G)$ of G. Moreover, if $f : G \longrightarrow K$ is a continuous locally definable map over A between connected locally definable groups over A with $f(e_G) = e_K$, then there is an induced homomorphim $f_* : \pi_1(G) \longrightarrow \pi_1(K)$ given by $f_*([\sigma]) = [f \circ \alpha]$. For details see [e2].

Proposition 3.5 *Let G and K be connected locally definable groups over A and let $f : G \longrightarrow K$ be a continuous locally definable map over A with $f(e_G) = e_K$. Then there is a canonical commutative diagram*

$$\begin{array}{ccc} \pi_1(G) & \xrightarrow{f_*} & \pi_1(K) \\ \downarrow & & \downarrow \\ H_1(G) & \xrightarrow{f_*} & H_1(K) \end{array}$$

in the category of groups. Moreover, the homomorphism $\pi_1(G) \longrightarrow H_1(G)$ induces the following isomorphism

$$\pi_1(G)/[\pi_1(G), \pi_1(G)] \longrightarrow H_1(G).$$

In fact, since by [e2], $\pi_1(G)$ is abelian, we have an isomorphism $\pi_1(G) \simeq H_1(G)$.

Proof. This the o-minimal Hurewicz theorem and it can be proved as its classical analogue. In fact, it is enough that on the proof of [ro] Lemma 4.26 we replace the function $u : [0,1] \longrightarrow \mathbb{S}^1$ given by $u(t) = e^{2\pi it}$ by any definable continuous map $v : [0,1] \longrightarrow \mathbb{S}^1$ such that $v(0) = v(1)$ and $v_{|[0,1)}$ is a bijection onto $\mathbb{S}^1$. □

Proposition 3.6 *If G is a locally definable group over A with dimension m, then $H_k(G) = 0$ for all $k > m$ and $H_m(G)$ is torsion-free.*

Proof. For $n \in \mathbb{Z}$, let $Z_n(G) = \mathrm{Ker}\partial_n$ and $B_n(G) = \mathrm{Im}\partial_{n+1}$. Then by definition we have $H_n(G) = Z_n(G)/B_n(G)$.

Suppose that $n > m$ and let $\sigma = \sum_{i=1}^{l} \sigma_i \in S_n(G)$, where each $\sigma_i : \Delta^n \longrightarrow G$ is definable (continuous) map, be such that $\sigma \in Z_n(G)$. By saturation, let U be an open definable subset of G such that $\cup\{\sigma_i(\Delta^n) : i = 1, \dots, l\} \subseteq U$. Then $\sigma \in Z_n(U)$. Thus, since $H_n(U) = 0$ for $n > \dim U = m$, there is $\lambda \in S_{n+1}(U)$ such that $\sigma = \partial_{n+1}\lambda$. But λ is also an element of $S_{n+1}(G)$ and, the equality $\sigma = \partial_{n+1}\lambda$ holds in $S_n(G)$. This shows that $Z_n(G) = B_n(G)$ as required.

Now suppose that $n = m$. Let $\sigma = \sum_{i=1}^{l} \sigma_i \in S_n(G)$, where each $\sigma_i : \Delta^n \longrightarrow G$ is definable (continuous) map, be such that $\sigma \in Z_n(G)$. Suppose also that there is $s \in \mathbb{N}$ such that $s\sigma = \partial_{n+1}\lambda$ for some $\lambda \in S_{n+1}(G)$, i.e., the class $[\sigma]$ of σ in $H_m(G)$ is a torsion element. By saturation, let U be an open definable subset of G such that $\cup\{\sigma_i(\Delta^n) : i = 1, \dots, l\} \cup \{\lambda_j(\Delta^{n+1}) : j = 1, \dots, k\} \subseteq U$ where $\lambda = \sum_{j=1}^{k} \lambda_j \in S_{n+1}(G)$ and each $\lambda_j : \Delta^{n+1} \longrightarrow G$ is definable (continuous) map. Then $\sigma \in Z_n(U)$ and $\lambda \in S_{n+1}(U)$. So the equality $s\sigma = \partial_{n+1}\lambda$ holds in $S_n(U)$. Thus the class $[\sigma]$ of σ in $H_m(U)$ is a torsion element which contradicts the fact that $H_m(U)$ is a (possibly trivial) free abelian group. □

3.3 Hopf algebras of locally definable groups

Let $H = \sum_{k \geq 0} H_k$ be a graded, skew-commutative, associative R-algebra with unity element where R is a commutative ring with unit. We say that H is *connected* if $H_0 \simeq R$.

If H is connected, we call H a *quasi Hopf R-algebra or hopf R-algebra* (of finite type) if each H_k is an R-module (resp., a finite dimensional R-module) and there is a degree preserving R-algebra homomorphism $\psi : H \longrightarrow H \otimes_R H$ called *co-multiplication or diagonal*, such that: if H_0 is an R-algebra of dimension one with generator e, then the map $\epsilon : H \longrightarrow R$, defined by $\epsilon(e) = 1$ and $\epsilon(h) = 0$ for all $h \in H_k$ with $k \geq 1$ is a *co-unit* i.e., for all $h \in H$, $(\epsilon \otimes_R 1)\psi(h) = 1 \otimes_R h$ and $(1 \otimes_R \epsilon)\psi(h) = h \otimes_R 1$. A quasi Hopf R-algebra H is called a *Hopf R-algebra* if ψ is associative i.e., $(\psi \otimes_R 1)\psi = (1 \otimes_R \psi)\psi$. We say that ψ is commutative if $T \circ \psi = \psi$ where $T(x \otimes_R y) = (-1)^{\deg(x)\deg(y)} y \otimes_R x$.

Example 3.7 An example of a connected Hopf R-algebra, is the *free, skew-commutative, graded Hopf R-algebra* $R[x_1, \ldots, x_k, \ldots] \otimes_R \bigwedge[y_1, \ldots, y_l, \ldots]_R$, where the x_i's are of even degrees and the y_j's are of odd degrees, and we have the relations: $y_j^2 = -y_j^2 = 0$, $y_i y_j = -y_j y_i$, $y_j x_i = x_i y_j$, $x_i x_j = x_j x_i$. In view of the freeness of this algebra, comultiplication is determined by its values on the generators x_i and y_j:

$$\psi(x_i) = x_i \otimes_R 1 + 1 \otimes_R x_i$$

and

$$\psi(y_j) = y_j \otimes_R 1 + 1 \otimes_R y_j.$$

This algebra is of finite type if the number of x_i's and y_j's of each degree is finite. For details on this example see [d] Chapter VII, Example 10.15.

We now recall the classical Hopf-Leray-Borel theorem classifying quasi Hopf algebras of finite type over perfect fields. For details see [mt] Chapter VII, Corollary 1.4.

Theorem 3.8 *Let H be a quasi Hopf algebra of finite type over a perfect field K_p of characteristic p. Then we have the following ring isomorphisms:*
(1) For $p = 0$; $H \simeq (\bigwedge_\alpha [x_\alpha]_{K_0}) \otimes (\bigotimes_\beta K_0[x_\beta])$, where $\deg x_\alpha$ is odd and $\deg x_\beta$ is even.
(2) For $p = 2$; $H \simeq (\bigotimes_\alpha K_2[x_\alpha]/(x_\alpha^{h_\alpha})) \otimes (\bigotimes_\beta K_2[x_\beta])$, where h_α is a power of 2.
(3) For $p \neq 0, 2$; $H \simeq (\bigwedge_\alpha [x_\alpha]_{K_p}) \otimes (\bigotimes_\beta K_p[x_\beta]) \otimes (\bigotimes_\gamma K_p[x_\gamma]/(x_\gamma^{h_\gamma})$, where $\deg x_\alpha$ is odd, $\deg x_\beta$ and $\deg x_\gamma$ are even, and h_γ is a power of p.

Here, if $\dim H < \infty$, then there is no term of $K_p[x_\beta]$.

In [d] Chapter VII, Proposition 10.16, it is proved that in the characteristic zero case, Theorem 3.8 holds for quasi Hopf algebras which are not necessarily of finite type. This is the original Hopf-Leray theorem in characteristic zero.

Theorem 3.9 *Let R be a field and G is a connected locally definable abelian group over A. Then $H_*(G;R)$ is a Hopf R-algebra such that $H_n(G;R)=0$ for all $n>\dim G$.*

Proof. The multiplication $\nu : H_*(G;R)\otimes_R H_*(G;R) \longrightarrow H_*(G;R)$ is called the Pontrijagin product and is defined by $\nu = u_* \circ \alpha''$ where $u : G\times G \longrightarrow G$ is the multiplication map on G and α'' is the cross product homomorphism in the Künneth formula for the o-minimal singular homology. By definition, ν preserves degrees. The associativity (resp., commutativity and units) for ν follows at once from the corresponding properties for u and α'' (Theorem 3.3). For details see [d] Chapter VII, Section 3. Since G is connected, by Propositions 3.1 and 3.4, $H_0(G;R)\simeq R$ and $H_*(G;R)$ is a connected, graded, skew-commutative, associative R-algebra with unity element.

The co-multiplication $\psi : H_*(G;R) \longrightarrow H_*(G;R)\otimes_R H_*(G;R)$ is given by $\psi = (\alpha'')^{-1}\circ \Delta_{G*}$ where $\Delta_G : G \longrightarrow G\times G$ is the diagonal and α'' is the cross product homomorphism in the Künneth formula for the o-minimal singular homology. A representative of ψ at the level of the o-minimal singular chain complexes is a natural diagonal map $D : EZ \circ \Delta_{G\sharp} : S_*(G) \longrightarrow S_*(G)\otimes S_*(G)$. The properties of the EZ-maps, like in the purely algebraic classical case, carry over to the diagonal D and we obtain that D is commutative, associative and preseves units as in [d] Chapter VI, (12.22) – (12.24). The fact that D is preserves units, i.e., $(\mathrm{id}\otimes\theta)\circ D$ and $(\theta\otimes\mathrm{id})\circ D$ are chain homotopic to id where $\theta : S_*(G) \longrightarrow \mathbb{Z}$ is the augmentation, corresponds exacts to the property that ψ needs to have to be a co-multiplication. Associativity (resp., commutativity) of D corresponds to the associativity (resp., commutativity) of ψ. For details see [d] Chapter VI, Section 12 and also [d] Chapter VII, Section 3 and Section 10.

By Proposition 3.6, $H_n(G)=0$ for all $n>\dim G$ and $H_n(G)$ is torsion-free for $n=\dim G$. Now Proposition 3.1 and [w] Proposition 3.1.4 implies that $H_n(G;R)=0$ for all $n>\dim G$. □

Corollary 3.10 *Let R be a field of characteristic zero and G a connected locally definable abelian group over A. Then*

$$H_*(G;R) \simeq \bigwedge[y_1,\ldots,y_r]_R$$

for some $r \leq \dim G$ such that $\sum_{i=1}^{r} \deg y_i \leq \dim G$.

Proof. This follows from the Hopf-Leray theorem in characteristic zero ([d] Chapter VII, Proposition 10.16) and Theorem 3.9. □

3.4 $H_*(G;\mathbb{Z}/p\mathbb{Z})$ for $p > \dim G + 1$

In this subsection we will prove a modification of the Hopf-Leray theorem classifying connected Hopf algebras over fields of characteristic zero and perfect fields of positive characteristic. We will follow the treatment of the characteristic zero case presented in [d] Chapter VII, Section 10. This modified version of the Hopf-Leray theorem will allow us to compute $H_*(G;\mathbb{Z}/p\mathbb{Z})$ for $p > \dim G + 1$.

Definition 3.11 Let $A = \sum_{k\geq 0} A_k$ be a connected, graded, skew-commutative associative R-algebra with unity element. Let $\mu : A \otimes_R A \longrightarrow A$ be the multiplication. We will often write $a_1 a_2$ for $\mu(a_1, a_2)$.

Define the graded R-submodules $D^n A$ of A where $n = 0, 1, \ldots$, as follows: $D^0 A = A$; $(D^1 A)_j = A_j$ for $j > 0$ and $(D^1 A)_j = 0$ for $j \leq 0$; and $D^{n+1} A = \mathrm{Im}(\mu : D^n A \otimes D^1 A \longrightarrow A)$. Clearly $D^{n+1} A \subseteq D^n A$ and $(D^n A)_j = 0$ for all $j < n$.

Define the R-modules $\Theta^n A = D^n A / D^{n+1} A$. We say that A is a *split R-algebra* if $D^{n+1} A$ is a direct summand of $D^n A$ for all n. For example, if R is a field, then A is always a split R-algebra.

Note that, by [d] Chapter VII, Proposition 10.4, D^n and Θ^n may be viewed as functors of connected, graded, skew-commutative, associative R-algebra with unity element. By [d] Chapter VII, (10.3) and Proposition 10.7, we have the following easy result.

Remark 3.12 Let A and B be connected, graded, skew-commutative, associative split R-algebra with unity element. Then $A \otimes_R B$ is also a connected, graded, skew-commutative, associative split R-algebra with unity element. Moreover, we have

$$D^n A \simeq \Theta^n A \oplus D^{n+1} A \simeq \bigoplus_{k \geq n} \Theta^k A,$$

$$D^n(A \otimes_R B) = \sum_{0 \leq i \leq n} D^i A \otimes_R D^{n-i} B$$

and

$$\Theta^n(A \otimes_R B) = \sum_{0 \leq i \leq n} \Theta^i A \otimes_R \Theta^{n-i} B.$$

Below, we will use $\pi_A^i : \Theta^n(A \otimes_R A) \longrightarrow \Theta^i A \otimes_R \Theta^{n-i} A$ to denote the natural projection and we will use $\iota_A^i : \Theta^i A \otimes_R \Theta^{n-i} A \longrightarrow \Theta^n(A \otimes_R A)$ to denote the natural inclusion.

Remark 3.13 Let $h : A \longrightarrow H$ be a homomorphism of connected, graded, skew-commutative, associative split R-algebras with unity element. Suppose that H is a Hopf R-algebra. Then the multiplication $\mu : A \otimes_R A \longrightarrow A$ of A and the comultiplication $\psi : H \longrightarrow H \otimes_R H$ of H are R-algebra homomorphisms. Moreover, if $\Theta^i h : \Theta^i A \longrightarrow \Theta^i H$ and $\Theta^{n-i} h : \Theta^{n-i} A \longrightarrow \Theta^{n-i} H$ are isomorphisms and

$$\theta_i^n h = (\Theta^n \mu) \circ \iota_A^i \circ (\Theta^i h \otimes_R \Theta^{n-i} h)^{-1} \circ \pi_H^i \circ (\Theta^n \psi) \circ (\Theta^n h),$$

then $\theta_i^n h : \Theta^n A \longrightarrow \Theta^n A$ agrees with the map $\frac{n!}{i!(n-i)!} \cdot 1 : \Theta^n A \longrightarrow \Theta^n A$.

For the proof of Remark 3.13, see [d] Chapter VII, Lemma 10.11. The next lemma is a modification of [d] Chapter VII, Proposition 10.14. The statement is different but the proof is essentially the same.

Lemma 3.14 *Let $p \geq 2$ be an integer. Let $h : A \longrightarrow H$ be a homomorphism of connected, graded, skew-commutative, associative split R-algebras with unity element. Suppose that H is a Hopf R-algebra and $D^1 A$ (as an abelian group) is k-torsion free for all $k < p$. If $\Theta^1 h : \Theta^1 A \longrightarrow \Theta^1 H$ is an isomorphism, then h is surjective and $h_j : A_j \longrightarrow H_j$ is an isomorphism for all $j \leq p - 2$.*

Proof. By [d] Chapter VII, Corollary 10.6, h is surjective. Since h is surjective, it follows easily from the definition that $D^n h : D^n A \longrightarrow D^n H$ is surjective for all n. But then, by the first equation on Remark 3.12, it follows that $\Theta^n h : \Theta^n A \longrightarrow \Theta^n H$ is also surjective for all n. We will show that $\Theta^n h : \Theta^n A \longrightarrow \Theta^n H$ is injective for all $1 \leq n < p$. Let $1 < n < p$. By induction assume that $\Theta^{n-1} h : \Theta^{n-1} A \longrightarrow \Theta^{n-1} H$ is an isomorphism. Then, by Remark 3.13, with $i = n - 1$, we see that $\theta^n_{n-1} h : \Theta^n A \longrightarrow \Theta^n A$ agrees with $n \cdot 1 : \Theta^n A \longrightarrow \Theta^n A$. By the first equation on Remark 3.12 and the assumption on $D^1 A$, we see that $\Theta^n A$ is n-torsion free. Hence, $\theta^n_{n-1} h$ is injective. In particular, the first factor $\Theta^n h$ in the composition defining the homomorphism $\theta^n_{n-1} h$ is injective.

By decreasing induction on $n < p$ we now show that $(D^n h)_j : (D^n A)_j \longrightarrow (D^n H)_j$ is an isomorphism for all $j \leq p - 2$. For $n = 0$ we get the lemma. Let $n = p - 1$. Since $j < n$, we have $(D^n A)_j = (D^n H)_j = 0$. On the other hand, by definition, we have the exact sequence $0 \to D^n A \to D^{n-1} A \to \Theta^{n-1} A \to 0$. In particular, for each $j \leq p - 2$, we have the exact sequence $0 \to (D^n A)_j \to (D^{n-1} A)_j \to (\Theta^{n-1} A)_j \to 0$. Now the inductive step follows from the exact sequence $0 \to (D^n A)_j \to (D^{n-1} A)_j \to (\Theta^{n-1} A)_j \to 0$ and the five lemma. □

We will now prove the following weak analogue of the Hopf-Leray-Borel theorem for Hopf algebras over $\mathbb{Z}/p\mathbb{Z}$ which are not necessarily of finite type.

Proposition 3.15 *Let $p \geq 2$ be a prime and let H be a Hopf $\mathbb{Z}/p\mathbb{Z}$-algebra. Then there is a free, skew-commutative, graded Hopf $\mathbb{Z}/p\mathbb{Z}$-algebra*

$$A = \mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_k, \ldots] \otimes_{\mathbb{Z}/p\mathbb{Z}} \bigwedge[y_1, \ldots, y_l, \ldots]_{\mathbb{Z}/p\mathbb{Z}}$$

and there is a homomorphism $h : A \longrightarrow H$ of connected, graded, skew-commutative, associative split $\mathbb{Z}/p\mathbb{Z}$-algebras with unity element such that h is surjective and $h_j : A_j \longrightarrow H_j$ is an isomorphism for all $j \leq p - 2$.

Proof. Since $\mathbb{Z}/p\mathbb{Z}$ is a field, the $\mathbb{Z}/p\mathbb{Z}$-module $\Theta^1 H = D^1 H / D^2 H$ has a basis over $\mathbb{Z}/p\mathbb{Z}$. Let $M \subseteq D^1 H \subseteq H$ be the lifting of a basis of $\Theta^1 H$ over $\mathbb{Z}/p\mathbb{Z}$. Let $A = \mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_k, \ldots] \otimes_{\mathbb{Z}/p\mathbb{Z}} \bigwedge[y_1, \ldots, y_l, \ldots]_{\mathbb{Z}/p\mathbb{Z}}$ where each x_i (resp., y_j) is in M and the degree of x_i (resp., y_j) in A is the same as its degree in H. Let $h : A \longrightarrow H$ be the unique homomorphism of connected, graded, skew-commutative, associative split $\mathbb{Z}/p\mathbb{Z}$-algebras with unity element extending the inclusion $M \longrightarrow H$.

The result will follow from Lemma 3.14. So we need to verify the conditions of the lemma. Clearly, the set $\{x + D^2A : x \in M\}$ is a basis for the $\mathbb{Z}/p\mathbb{Z}$-module $\Theta^1 A = D^1A/D^2A$. Hence, $\Theta^1 h : \Theta^1 A \longrightarrow \Theta^1 H$ is an isomorphism.

We will show that D^1A is k-torsion free for every $k < p$. Let $0 < k < p$ and let $a \in D^1A$ be an element such that $ka = 0$. Since the elements of the form $x_{i_1} \cdot \cdots \cdot x_{i_l} \otimes_{\mathbb{Z}/p\mathbb{Z}} y_{j_1} \wedge \cdots \wedge y_{j_n}$ with $j_1 < \ldots < j_n$ are a basis for A over $\mathbb{Z}/p\mathbb{Z}$, we have that a is a finite sum $\sum_{i_1,\ldots,i_l,\, j_1<\ldots<j_n} r^{j_1,\ldots,j_n}_{i_1,\ldots,i_l} x_{i_1} \cdot \cdots \cdot x_{i_l} \otimes_{\mathbb{Z}/p\mathbb{Z}} y_{j_1} \wedge \cdots \wedge y_{j_n}$. Therefore, $ka = \sum_{i_1,\ldots,i_l,\, j_1<\ldots<j_n} kr^{j_1,\ldots,j_n}_{i_1,\ldots,i_l} x_{i_1} \cdot \cdots \cdot x_{i_l} \otimes_{\mathbb{Z}/p\mathbb{Z}} y_{j_1} \wedge \cdots \wedge y_{j_n} = 0$. But this implies that $kr^{j_1,\ldots,j_n}_{i_1,\ldots,i_l}$ is zero in $\mathbb{Z}/p\mathbb{Z}$ and therefore, all the elements $r^{j_1,\ldots,j_n}_{i_1,\ldots,i_l}$ are zero in $\mathbb{Z}/p\mathbb{Z}$. So a is the zero element. □

Corollary 3.16 *Let G be a connected locally definable abelian group over A and $p > \dim G + 1$ a prime. Then*

$$H_*(G; \mathbb{Z}/p\mathbb{Z}) \simeq \bigwedge [y_1, \ldots, y_t]_{\mathbb{Z}/p\mathbb{Z}}$$

for some $t \leq \dim G$ with $\sum_{i=1}^{t} \deg y_i \leq \dim G$.

Proof. This is a consequence of Proposition 3.15 and Theorem 3.9. □

We end with the following conjecture whose o-minimal cohomology analogue was verified in [eo] for definable groups.

Conjecture: If G is a connected locally definable abelian group over A and K_p is a perfect field of characteristic p (a prime or zero), then

$$H_*(G; K_p) = \bigwedge [y_{1:1}, \ldots, y_{m:1}]_{K_p}$$

with $\deg y_{l:1} = 1$ for all $l = 1, \ldots, m$. Moreover, $m = \dim G$ if and only if G is a definably compact definable abelian group.

4 Proof of the main theorems

We now use the Hopf algebra of a connected locally definable abelian group G over A to extract information about its o-minimal fundamental group.

Lemma 4.1 *Let G be a connected locally definable abelian group over A. Then the following hold.*

(1) For every prime number p, the subgroup $\pi_1(G)[p]$ of p-torsion points of $\pi_1(G)$ is finite with at most $\dim G$ generators.

(2) For every prime number p and natural number k, the subgroup $\pi_1(G)[p^k]$ of p^k-torsion points of $\pi_1(G)$ is finite with at most $\dim G$ generators.

(3) The subgroup $\pi_1(G)[\infty]$ of torsion free points of $\pi_1(G)$ is finitely generated with at most $\dim G$ generators.

Proof. Fisrt observe that, by Proposition 3.5, $\pi_1(G) \simeq H_1(G)$. The arguments in the proofs of (1) and (3) are similar and (2) follows from (1).

(1) We have $H_1(G;\mathbb{Z}/p\mathbb{Z}) \simeq H_1(G) \otimes \mathbb{Z}/p\mathbb{Z} \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_0(G), \mathbb{Z}/p\mathbb{Z})$ by Proposition 3.1. As G is connected, by Proposition 3.4, $H_0(G) \simeq \mathbb{Z}$ and so, $H_1(G;\mathbb{Z}/p\mathbb{Z}) \simeq H_1(G) \otimes \mathbb{Z}/p\mathbb{Z} = \mathrm{Tor}_0^{\mathbb{Z}}(H_1(G), \mathbb{Z}/p\mathbb{Z}) \simeq H_1(G)[p] \simeq \pi_1(G)[p]$. For details, see [w] 3.1.1.

By Corollary 3.16 the Hopf $\mathbb{Z}/p\mathbb{Z}$-algebra $H_*(G;\mathbb{Z}/p\mathbb{Z})$ is isomorphic to $\bigwedge[y_1, \ldots, y_t]_{\mathbb{Z}/p\mathbb{Z}}$ with $t \leq \dim G$ such that $\sum_{i=1}^{t} \deg y_i \leq \dim G$. Hence, since a generator of $\pi_1(G)[p]$ determines a generator of $H_*(G;\mathbb{Z}/p\mathbb{Z})$ of degree one, the group $\pi_1(G)[p]$ has at most $\dim G$ generators.

(2) Let $x_1, \ldots, x_m \in \pi_1(G)[p^k]$ be such that, for each $i = 1, \ldots, m$, x_i is not in the subgroup of $\pi_1(G)[p^k]$ generated by $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m\}$. Then the subgroup of $\pi_1(G)[p^k]$ generated by $\{x_1, \ldots, x_m\}$ is isomorphic to $\bigoplus_{i=1}^{m} \mathbb{Z}/p^{k_i}\mathbb{Z}$ and, consequently, $\pi_1(G)[p]$ contains a subgroup with m generators. Thus by (1) $m \leq \dim G$ and (2) holds.

(3) By Proposition 3.1, $H_1(G;\mathbb{Q}) \simeq H_1(G) \otimes \mathbb{Q} \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_0(G), \mathbb{Q})$. As G is connected, by Proposition 3.4, $H_0(G) \simeq \mathbb{Z}$ and so, $H_1(G;\mathbb{Q}) \simeq H_1(G) \otimes \mathbb{Q} \simeq H_1(G)[\infty] \otimes \mathbb{Q} \simeq \pi_1(G)[\infty] \otimes \mathbb{Q}$.

By Corollary 3.10, $H_*(G;\mathbb{Q}) \simeq \bigwedge[y_1, \ldots, y_m]_{\mathbb{Q}}$ with $m \leq \dim G$ such that $\sum_{i=1}^{m} \deg y_i \leq \dim G$. Hence, since a generator of $\pi_1(G)[\infty]$ determines a generator of $H_*(G;\mathbb{Q})$ of degree one, the group $\pi_1(G)[\infty]$ has at most $\dim G$ generators. □

Proof of Theorem 1.1. Let G be a locally definable abelian group over A and let m be a natural number with prime decomposition $p_1^{k_1} \cdot \dots \cdot p_l^{k_l}$ such that $p_1, \dots, p_l \geq \dim G + 2$. We have to show that the subgroup $G[m]$ of m-torsion points of G is a finite locally definable subgroup over A with at most $2l \dim G$ generators. Since $G[m] \simeq G[p^{k_1}] \oplus \dots \oplus G[p^{k_l}]$ it is enough to show that if p is a prime such that $p \geq \dim G + 2$, then for all $k \in \mathbb{N}$, the subgroup $G[p^k]$ of p^k-torsion points of G is a finite locally definable subgroup over A with at most $2 \dim G$ generators.

For a contradiction suppose that there are distinct elements $x_1, \dots, x_m \in G[p]$ with $m > 2 \dim G$ such that, for each $i = 1, \dots, m$, x_i is not in the subgroup of $G[p]$ generated by $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m\}$.

Let Z be the subgroup of G generated by $\{x_1, \dots, x_m\}$. Then Z is a finite group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ and hence Z is a locally definable subgroup of G over A of dimension zero. By (QT) and [e2] Theorem 3.5, we have a locally definable covering homomorphism $h : G \longrightarrow G/Z$. By [e2] Propositions 3.4 and 4.6, we have a short exact sequence

$$1 \to h_*(\pi_1(G)) \to \pi_1(G/Z) \stackrel{\psi}{\to} Z \to 1.$$

For each $i = 1, \dots, m$, choose $u_i \in \pi_1(G/Z)$ is such that $\psi(u_i) = x_i$. Note that we can choose $u_1, \dots, u_m$ so that for each $i = 1, \dots, m$, the element u_i is not in the subgroup of $\pi_1(G/Z)$ generated by $\{u_1, \dots, u_{i-1}\}$. Observe also that either u_i is torsion free or the order l_i of u_i is divisible by p. Indeed, if $ku_i = 0$, then $0 = \psi(ku_i) = k\psi(u_i) = kx_i$. So, since $m > 2 \dim G = \dim G/Z$, either there are more than $\dim G/Z$ elements of $\{u_1, \dots, u_m\}$ which are generators of $\pi_1(G/Z)[\infty]$ or there are more than $\dim G/Z$ elements $w_1, \dots, w_n$ in $\{u_1, \dots, u_m\}$ of finite order divisible by p. The first case cannot happen by Lemma 4.1 (2). So assume that the later holds. For each $i = 1, \dots, n$, write $l_i = p^{k_i} s_i$ with s_i not divisible by p. Then the finite subgroup generated by w_i is of the form $\mathbb{Z}/p^{k_i}\mathbb{Z} \times F_i$ with F_i of order s_i. Since s_i is prime to p, F_i is in the kernel of ψ. So we may replace each w_i by an element v_i of order p^{k_i}. Consequently, $\pi_1(G/Z)[p^k]$ where $k = k_1 + \dots + k_n$, contains $n > \dim G/Z$ generators which contradicts Lemma 4.1 (2).

Let $x_1, \dots, x_m \in G[p^k]$ be such that, for each $i = 1, \dots, m$, x_i is not in the subgroup of $G[p^k]$ generated by $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m\}$. Then the subgroup of $G[p^k]$ generated by $\{x_1, \dots, x_m\}$ is isomorphic to $\bigoplus_{i=1}^m \mathbb{Z}/p^{k_i}\mathbb{Z}$

and, consequently, $G[p]$ contains a subgroup with m generators. Thus $m \leq \dim G$ and $G[p^k]$ is a finite locally definable subgroup over A with at most $2 \dim G$ generators. $\square$

5 Concluding remarks

The goal of this section is to present a proof of o-minimal homology structure theorem for definably compact definable abelian groups (Theorem 1.5). However, when possible, we will develop the theory in the category of locally definable groups rather than just definable groups.

We start with the generalization of the Euler characteristic. So let G be a connected locally definable abelian group. Then by Corollary 3.10, we have $H_*(G;\mathbb{Q}) \simeq \bigwedge[y_1,\dots,y_r]_{\mathbb{Q}}$ for some $r \leq \dim G$ such that $\sum_{i=1}^{r} \deg y_i \leq \dim G$. The *Euler-Poincaré characteristic* $\chi(G)$ of G is by definition

$$\chi(G) = \sum_{i=0}^{\dim G} (-1)^i b_i$$

where $b_i = \dim_{\mathbb{Q}} H_i(G;\mathbb{Q})$ are the Betti numbers of G.

This is indeed a generalization of the o-minimal Euler characteristic for definable groups, since if G is a definably compact abelian group, then by [bo], $\chi(G)$ coincides with the o-minimal Euler characteristic $E(G)$ of G.

If $f : G \longrightarrow G$ is a continuous locally definable map, the *Lefschetz number of f* is defined by

$$\lambda(f) = \sum_{i=0}^{\dim G} (-1)^i \mathrm{tr}(f_*)_i$$

where $\mathrm{tr}(f_*)_i$ is the trace of $(f_*)_i : H_i(G;\mathbb{Q}) \longrightarrow H_i(G;\mathbb{Q})$.

Theorem 5.1 *Let G be a connected locally definable abelian group. For each $k \in \mathbb{N}$, let $[k] : G \longrightarrow G$ be the locally definable homomorphism given by multiplication by k. Then the following hold.*

(1) If $x_i = y_{i_1} \wedge \cdots \wedge y_{i_u}$ is a generating monomial for the graded vector space $H_*(G;\mathbb{Q}) \simeq \bigwedge[y_1,\dots,y_r]_{\mathbb{Q}}$, then $[k]_*(x_i) = k^{\mathrm{len}(x_i)} x_i$ where $\mathrm{len}(x_i) = u$.

(2) If $r > 0$, then $\lambda([k]) = (1-k)^r$. In particular, $\chi(G) = 0$.

Proof. By Corollary 3.10, $H_*(G;\mathbb{Q}) \simeq \bigwedge[y_1,\ldots,y_r]_{\mathbb{Q}}$ for some $r \leq \dim G$ such that $\sum_{i=1}^r \deg y_i \leq \dim G$. Therefore, co-multiplication is given by $\mu(x) = x \otimes_R 1 + 1 \otimes_R x$ for all $x \in \{y_1,\ldots,y_r\}$. Since by Theorem 3.9, $\mu = (\alpha'')^{-1} \circ \Delta_{G*}$ (where $\Delta_G : G \longrightarrow G \times G$ is the diagonal map and α'' is cross product homomorphism in the Künneth formula for the o-minimal singular homology), we have $\Delta_{G*}(x) = x \times 1 + 1 \times x$ for all $x \in \{y_1,\ldots,y_r\}$. Recall also that the product in $H_*(G;\mathbb{Q})$ is given by $u_* \circ \alpha''$ where $u : G \times G \longrightarrow G$ is the product in G.

(1) An induction on k shows that $[k]_*(x) = kx$ for $x \in \{y_1,\ldots,y_r\}$. In fact, $[k+1] = u \circ ([k] \times 1_G) \circ \Delta_G$ and so $[k+1]_*(x) = u_* \circ ([k] \times 1_G)_* \circ \Delta_{G*}(x) = u_* \circ ([k] \times 1_G)_*(x \times 1 + 1 \times x)$. By the naturality of cross product and the inductive hypothesis, we have $[k+1]_*(x) = u_*(kx \times 1 + 1 \times x) = u_* \circ \alpha''(kx \otimes 1 + 1 \otimes x) = (k+1)x$. Therefore, if $x_i = y_{i_1} \wedge \cdots \wedge y_{i_u}$ is a generating monomial for the graded vector space $H_*(G;\mathbb{Q})$, then $[k]_*(x_i) = k^{\mathrm{len}(x_i)} x_i$ where $\mathrm{len}(x_i) = u$.

We now prove condition (2). By (1), $\lambda([k]) = \sum (-1)^{\deg(x_i)} k^{\mathrm{len}(x_i)} + 1$ where the sum is taken over all generating monomials $x_i = y_{i_1} \wedge \cdots \wedge y_{i_u}$ for the graded vector space $H_*(G;\mathbb{Q})$. Since y_j's have odd degrees, we have $(-1)^{\deg(x_i)} = (-1)^{\mathrm{len}(x_i)}$. Using this, a simple calculation shows that $\lambda([k]) = (1-k)^r$. □

Theorem 5.1 together with the basic theory of Subsections 3.1 and 3.3 gives an easy solution to the Peterzil-Steinhorn torsion points problem ([ps]):

Remark 5.2 (Torsion points problem) If G is a definably connected definably compact abelian group of dimension n, then G has m-torsion points for every $m \in \mathbb{N}$.

In fact, by [bo], $H_n(G) \simeq \mathbb{Z}$ and so, by the universal coefficients theorem, $H_n(G;\mathbb{Q}) \simeq \mathbb{Q}$. Hence, from Theorem 5.1 (2), it follows that $E(G) = 0$. But by [s], the condition $E(G) = 0$ implies that G has m-torsion points for every $m \in \mathbb{N}$.

Proof of Theorem 1.5. To prove Theorem 1.5 we require the theory of degrees of continuous definable maps as developed in [eo] Section 4. This theory is not available in the category of locally definable spaces.

Assume G is a definably connected, definably compact definable abelian group of dimension n and, for $m \in \mathbb{N}$, let $[m] : G \longrightarrow G$ be the definable homomorphism given by multiplication by m. by [bo], $H_n(G) \simeq \mathbb{Z}$ and so, by the universal coefficients theorem and Corollary 3.10, $H_*(G;\mathbb{Q}) \simeq \bigwedge[y_1,\dots,y_r]_{\mathbb{Q}}$ for some $0 < r \leq \dim G$ such that $\sum_{i=1}^{r} \deg y_i \leq \dim G$. By Theorem 5.1 (1), we have $[m]_*(\zeta_G) = m^r\zeta_G$ where $\zeta_G = y_1 \wedge \cdots \wedge y_r \in H_n(G;\mathbb{Q})$ is the fundamental class of G. Thus, by definition of degree of a definable continuous map, we have $\deg[m] = m^r$. By [eo] Corollary 4.6, $\deg[m] \leq |G[m]|$, and so, by Theorem 1.4, $r \leq s$ where s is such that $\pi_1(G) \simeq \mathbb{Z}^s$. Now Proposition 3.5 (or its definable analogue [eo] Theorem 5.1), $\pi_1(G) \simeq H_1(G)$, and the universal coefficients theorem, implies that among $\{y_1,\dots,y_r\}$ must be exactly s elements of degree one. Hence $s = r$ and all y_i's must be of degree one. Since $\zeta_G = y_1 \wedge \cdots \wedge y_r \in H_n(G;\mathbb{Q})$, it follows that $s = r = n$.

References

[bo] A.Berarducci and M.Otero *O-minimal fundamental group, homology and manifolds* J. London Math. Soc. (2) 65 (2002) 257–270.

[dk] H.Delfs and M.Knebusch *Locally semi-algebraic spaces* Springer Verlag 1985.

[d] A.Dold *Lectures on algebraic topology* Springer Verlag 1980.

[vdd] L. van den Dries, *Tame Topology and o-minimal structures* Cambridge University Press 1998.

[e1] M.Edmundo *Solvable groups definable in o-minimal structures* J. Pure Appl. Algebra 185 (2003) 103–145.

[e2] M.Edmundo *Covers of groups definable in o-minimal structures* Illinois J. Math ?? (2005) ??-??.

[e3] M.Edmundo *Locally definable groups in o-minimal structures* J.Algebra ?? (2005) ??-??.

[eo] M.Edmundo and M.Otero *Definably compact abelian groups* J. Math. Logic 4 (2) (2004) 163–180.

[ks] J.Krajcek and T. Scanlon *Combinatorics with definable sets: Euler characteristics and Grothendieck rings* Bull. Symbolic Logic 6 (2000) 311–330.

[mt] M.Mimura and H.Toda *Topology of Lie groups, I and II* American Mathematical Society 1991.

[pst] Y.Peterzil and S.Starchenko *Definable homomorphisms of abelian groups definable in o-minimal structures* Ann. Pure Appl. Logic 101 (1) (1999) 1–27.

[ps] Y.Peterzil and C.Steinhorn *Definable compacteness and definable subgroups of o-minimal groups* J. London Math. Soc. 59 (2) (1999) 769–786.

[pps] Y.Peterzil, A.Pillay and S.Starchencko *Simple algebraic groups over real closed fields* Trans. Amer. Math. Soc. 352 (10) (2000) 4421–4450.

[p] A.Pillay *On groups and fields definable in o-minimal structures* J. Pure Appl. Algebra 53 (1988) 239–255.

[ro] J.Rotman *An introduction to algebraic topology* Springer Verlag 1988.

[s] A.Strzebonski *Euler characteristic in semialgebraic and other o-minimal groups* J. Pure Appl. Algebra 96 (1994) 173–201.

[w] C.Weibel *An introduction to homological algebra* Cambridge University Press 1994.

[Wo] A.Woerheide *O-minimal homology* PhD. Thesis (1996), University of Illinois at Urbana-Champaign.