A uniform bound for torsion points in o-minimal expansions of groups

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Abstract

Here we prove the proper base change theorem and the projection formula for sheaf cohomology theory in o-minimal expansions of groups. As a consequence of this we prove in this setting the Künneth formula. These results when applied to definable groups give a uniform bound on the size of the torsion subgroups. We also prove these results in arbitrary o-minimal structures with definable Skolem functions.

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1 Introduction

Let $\mathcal{R}$ be an o-minimal expansion of an ordered group $(R, 0, +, <)$. The structure $\mathcal{R}$ will be fixed throughout and definable will mean definable in $\mathcal{R}$ with parameters.

In the paper [15] we constructed an o-minimal sheaf cohomology theory for the category of definable sets equipped with the o-minimal site and with morphisms continuous definable maps. This sheaf cohomology theory satisfies the Eilenberg-Steenrod axioms adapted to the o-minimal site. The o-minimal setting generalises the semi-algebraic and sub-analytic contexts ([4] and [11]), and so this theory generalises the existence of sheaf cohomology in semi-algebraic geometry, as described in the book [9]. See also [8].

Unlike in the case of o-minimal expansions of fields where we know that definable groups equipped with their definable manifold structure are affine, in order to apply this cohomology theory to definable groups in $\mathcal{R}$ we have to extend this theory to definable spaces. In Section 2 we point out that there is no problem to do this.

We are interested in applying this cohomology theory to definably compact definable groups. So we will work with definably compact spaces. Recall that a definable space X is called *definably compact* if for every continuous definable map $\alpha : (a, b) \subseteq R \longrightarrow X$, the limits $\lim_{t \longrightarrow a^+} \alpha(t)$ and $\lim_{t \longrightarrow b^-} \alpha(t)$ exist in X. See [26].

In the paper [2], it is proved that for a definably compact definable set $X \subseteq R^n$ and an abelian group F, the sheaf cohomology groups $H^p(X; F)$ with coefficients in the constant sheaf on X generated by F are finitely generated and invariant in elementary extensions of $\mathcal{R}$. In Section 2 we observe that the proof in [2] actually easily generalizes to definably compact definable spaces. This invariance result together with results from [15] will give us the proper base change theorem (Theorem 2.4). When this is combined with the projection formula (Theorem 2.7) we get the Künneth formula (Corollary 2.11). These results when applied to definable groups give a uniform bound on the size of the torsion subgroups:

Theorem 1.1 *Let G be a definably connected, definably compact, definable abelian group. Then there exists $s \leq \dim G$ such that the o-minimal fundamental group of G is*

$$\pi_1(G) \simeq \mathbb{Z}^s$$

and hence the subgroup of k-torsion points of G is

$$G[k] \simeq (\mathbb{Z}/k\mathbb{Z})^s.$$

After a preliminary version of the paper was in circulation, Y. Peterzil produced a paper ([25]) where is it proved by reduction of the semi-bounded case to the field case that s in Theorem 1.1 is actually equal to dim$G$. Also A. Berarducci produced a paper ([1]) with results on the o-minimal cohomology of definable groups in o-minimal expansions of fields which presents a lemma, due to A. Fornasiero, which can be used to prove the Künneth formula for o-minimal sheaf cohomology in arbitrary o-minimal structures with definable Skolem functions. With this result available together with the Hurewicz theorem from [16] we can prove a generalization of Theorem 1.1:

Theorem 1.2 *Let $\mathcal{M}$ be an arbitrary o-minimal structure with definable Skolem functions. Let G be a definably connected, definably compact definable abelian group. Then the classifying group of G is given by*

$$\eta(G) \simeq \Pi_{p \text{ prime}} \Pi_{\kappa(p)} \mathbb{Z}_p$$

where for each p, $\mathbb{Z}_p$ is the group of p-adic integers and if $p > \dim G + 1$, then $\kappa(p) \leq \dim G$. Hence, for each prime $p > \dim G + 1$, the subgroup of p-torsion points of G is

$$G[p] \simeq (\mathbb{Z}/p\mathbb{Z})^{\kappa(p)}$$

with $\kappa(p) \leq \dim G$.

We added an appendix to the paper explaining how to obtain Theorem 1.2.

2 Proper base change

As defined in [11] Chapter X, a *definable space* is a triple $(X, (X_i, \phi_i)_{i=1}^k)$ where: (i) $X = \cup\{X_i : i = 1, \ldots, k\}$; (ii) each $\phi_i : X_i \longrightarrow R^{l_i}$ is a bijection such that $\phi_i(X_i)$ is a definable subset of R^{l_i} and, for all j, $\phi_i(X_i \cap X_j)$ is open in $\phi_i(X_i)$ and the transition maps $\phi_{ij} : \phi_i(X_i \cap X_j) \longrightarrow \phi_j(X_i \cap X_j) : x \mapsto \phi_j(\phi_i^{-1}(x))$ are definable homeomorphisms. A definable space has a topology such that each X_i is open and the ϕ_i's are homeomorphisms: a subset U of X is an open in the basis for this topology if and only if for each i, $\phi_i(U \cap X_i)$ is an open definable subset of $\phi_i(X_i)$. We also say that a subset A of X is definable if and only if for each i, $\phi_i(A \cap X_i)$ is a definable subset of $\phi_i(X_i)$. A map between definable spaces is definable if when it is read through the charts it is definable. Thus we have the category of definable spaces with definable continuous maps. We also have the category of pairs of definable spaces whose objects are pairs (X, A) with X a definable space and A a definable subset of X and whose morphisms $f : (X, A) \longrightarrow (Y, B)$

are continuous definable maps $f : X \longrightarrow Y$ between definable spaces such that $f(A) \subseteq B$.

The *o-minimal site* on a definable space X will be the category whose objects are open definable subsets of X, the morphisms are the inclusions and the admissible covers are finite covers by open definable subsets. Thus we have the category of sheaves of abelian groups $\mathrm{Sh}_{\mathrm{dtop}}(X)$ on a definable space X equipped with the o-minimal site.

We can also define the *o-minimal spectrum* $\widetilde{X}$ of a definable space X as in the case of definable sets ([7] and [28]): it is the set of ultrafilters of definable subsets of X. The o-minimal spectrum $\widetilde{X}$ of a definable space X is a spectral topological space when equipped with the topology generated by the open subsets of the form $\widetilde{U}$, where U is an open definable subset of X. The proof of this is as in [15] Proposition 2.5 by reading through the charts. Also as in [15] Definition 2.8 we have the o-minimal spectrum $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ of a continuous definable map $f : X \longrightarrow Y$ between definable spaces. Hence we can define as in [15] sheaf cohomology in the category of pairs of definable spaces by setting

$$H^*(X; \mathcal{F}) := H^*(\widetilde{X}; \widetilde{\mathcal{F}})$$

where X is a definable space, $\mathcal{F}$ is a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$. (In this context we also have an analogue of [15] Proposition 3.2 giving an isomorphism of categories $\mathrm{Sh}_{\mathrm{dtop}}(X) \longrightarrow \mathrm{Sh}(\widetilde{X}) : \mathcal{F} \mapsto \widetilde{\mathcal{F}}$, where $\mathrm{Sh}(\widetilde{X})$ is the category of sheaves of abelian groups on the topological space $\widetilde{X}$).

We say that a definable space X is *definably normal* if for every two disjoint closed definable subsets A and B of X there are two disjoint open definable subsets U and W of X such that $A \subseteq U$ and $B \subseteq W$. Since by [11] Chapter VI, Proposition 1.2, $\mathcal{R}$ has definable Skolem functions, then by [11] Chapter VI, Proposition 1.14 and the proof of the main result from [15] we have that: In the category of pairs of definable spaces in $\mathcal{R}$ there exists an o-minimal sheaf cohomology theory satisfying the Eilenberg-Steenrod axioms.

The results from [2] that we mentioned in the introduction are based on cell decomposition for definable sets, thus we need a similar notion for definable spaces.

Let $(X, (X_i, \phi_i)_{i=1}^k)$ be a definable space. Define inductively the following data: K_1 is a cell decomposition of $\phi_1(X_1)$; if K_i has been defined, define K_{i+1} to be a cell decomposition of $\phi_{i+1}(X_{i+1})$ compatible with $\phi_{i+1}(X_{i+1}) \setminus \cup\{\phi_j(X_j \cap X_{i+1}) : j = 1, \dots, i\}$ and with the collection of definable sets $\{\phi_{i+1}(\phi_j^{-1}(C)) : j = 1, \dots, i,\ C \in K_j\}$. We call the data $(K_i)_{i=1}^k$ a *cell decomposition* of X and we call each $\phi_j^{-1}(C)$ with $C \in K_j$ for some j a *cell* in X of the cell decomposition $(K_i)_{i=1}^k$.

We say that a cell $\phi_j^{-1}(C)$ in a cell decomposition $(K_i)_{i=1}^k$ of a definable space X is bounded if C is a bounded cell in R^{l_j}. We say that a definable space X is *bounded* if each chart X_i is a bounded definable subset of R^{l_i}. So in a bounded definable space every cell is bounded.

With these definitions available we have the analogue of Lemmas 8.1 and 8.2 in [2] for bounded cells in definable spaces by reading through the charts. Also, if X is a bounded, definably compact, definably normal definable space, then we also have an analogue of Corollary 8.4 from [2] since Corollary C.6 there also holds for definable spaces - by replacing the use of Corollary C.5 in [2] by its generalization to arbitrary o-minimal structures:

Proposition 2.1 *Assume that X is a subspace of a normal space in the category of o-minimal spectra of definable spaces, $\mathcal{F}$ is a sheaf in* $\mathrm{Sh}(X)$ *and Y is a quasi-compact subset of X. Then the canonical homomorphism*

$$\lim_{Y\subseteq U, U \text{ open in } X} H^q(U;\mathcal{F}) \longrightarrow H^q(Y;\mathcal{F}_{|Y})$$

is an isomorphism for every $q \geq 0$.

As we point out in [15] page 175, this proposition is the o-minimal analogue of [8] Theorem 3.1 and follows in the same way from the shrinking lemma (Proposition 2.17 in [15]). Therefore we have:

Theorem 2.2 *Let X be a bounded, definably compact, definably normal definable space and let F be an abelian group. Then the sheaf cohomology groups $H^p(X;F)$ with coefficients in the constant sheaf on X generated by F are finitely generated and $H^p(X;F) = 0$ for $p >$ $\dim X$.*

As in [2] the proof of Theorem 2.2 also shows that:

Theorem 2.3 *Let X be a bounded, definably compact, definably normal definable space and let F be an abelian group. If $\mathcal{S}$ is an elementary extension of $\mathcal{R}$, then the restriction map $\widetilde{X(S)} \longrightarrow \widetilde{X}$ induces an isomorphism of sheaf cohomology*

$$H^*(\widetilde{X};F) \longrightarrow H^*(\widetilde{X(S)};F)$$

with coefficients in the constant sheaf generated by F.

We will now apply the invariance theorem (Theorem 2.3) to prove an o-minimal proper base change theorem which generalizes the proper base change from real algebraic geometry ([9] Chapter II, Theorem 7.8) but restricted to constant sheaves and bounded, definably compact, definably normal definable spaces.

Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be continuous definable maps between bounded, definably compact, definably normal definable spaces. Form the cartesian square

$$\begin{array}{ccc} X & \stackrel{g'}{\longleftarrow} & X \times_Y Z \\ \downarrow^{f} & & \downarrow^{f'} \\ Y & \stackrel{g}{\longleftarrow} & Z. \end{array}$$

Let $\mathcal{F}$ be a sheaf on X with respect to the o-minimal site. Then there is a canonical base change homomorphism

$$\alpha : g^* R^q f_* \mathcal{F} \longrightarrow R^q f'_* (g'^* \mathcal{F})$$

for every $q \geq 0$.

Theorem 2.4 (Proper base change) *If F is an abelian group, then the base change homomorphism*

$$\alpha : g^* R^q f_* F \longrightarrow R^q f'_* (g'^* F)$$

is an isomorphism for every $q \geq 0$.

Proof. Since $(g')^{-1}(A) = A \times_Y Z$, $f'_{|(g')^{-1}(A)}$ is definably proper for every closed definable subset A. This implies that $f'(B)$ is a closed definable subset of Z for every closed definable subset B. Also since X, Y and Z are definably normal, we see that $\widetilde{f'}$ (and of course $\widetilde{f}$) satisfy the hypothesis of following claim:

Claim 2.5 *Let $f : X \longrightarrow Y$ be a morphism in the category of o-minimal spectra of definable spaces. Assume that f maps constructible closed subsets of X onto closed subsets of Y. Let $\mathcal{F}$ be a sheaf in* $\mathrm{Sh}(X)$ *and suppose that Y is a subspace of a normal space in the category of o-minimal spectra of definable spaces. Then, for every $\beta \in Y$, the canonical homomorphism*

$$(R^q f_* \mathcal{F})_\beta \longrightarrow H^q(f^{-1}(\beta); \mathcal{F}_{|f^{-1}(\beta)})$$

is an isomorphism, where $(R^q f_* \mathcal{F})_\beta$ denotes the stalk of the higher direct image $R^q f_* \mathcal{F}$ at β.

As we point out in [15] page 175, this claim is the o-minimal analogue of [8] Theorem 3.5 and follows in the same way from Proposition 2.20 in [15].

Let $\gamma \in \widetilde{Z}$ and $\beta = \widetilde{g}(\gamma)$. From Claim 2.5 we have

$$(\widetilde{g}^* R^q \widetilde{f}_* F)_\gamma = (R^q \widetilde{f}_* F)_\beta = H^q(\widetilde{f}^{-1}(\beta); F)$$

and

$$(R^q\widetilde{f'}_*(\widetilde{g'}^*F))_\gamma = H^q(\widetilde{f'}^{-1}(\gamma);F).$$

Let $\mathcal{S}$ be an elementary extension of $\mathcal{R}$ where β is realized. Then we have $\widetilde{f^{-1}(\beta)}(S) = \widetilde{(f^S)^{-1}(\beta)}(S)$ and g' induces a homeomorphism between $\widetilde{f'}^{-1}(\gamma)$ and $\widetilde{(f^S)^{-1}(\beta)}(S)$. Hence

$$H^*(\widetilde{f'}^{-1}(\gamma);F) \simeq H^*(\widetilde{f^{-1}(\beta)}(S);F).$$

The result will follow from the next claim since then we get

$$H^*(\widetilde{f'}^{-1}(\gamma);F) \simeq H^*(\widetilde{f}^{-1}(\beta);F)$$

which shows that the base change homomorphism α is an isomorphism.

Claim 2.6 *We have an isomorphism*

$$H^*(\widetilde{f^{-1}(\beta)}(S);F) \simeq H^*(\widetilde{f^{-1}(\beta)};F).$$

We now prove the claim. Consider the cover of Y by its charts. Since Y is definably normal, by the shrinking lemma ([15] Proposition 2.17), we find a cover of Y by closed definable neighborhoods in Y each of which is contained in a chart of Y. Let O be one of these closed definable neighborhoods in Y such that $\beta \in \widetilde{O}$. By [20] and working through the charts, the specializations of β in $\widetilde{O}$ form a chain and hence there are only finitely many of them. Then there is an open definable subset U of O such that $\beta \in \widetilde{U}$ and no proper specialization of β in $\widetilde{O}$ is in $\widetilde{U}$ (remove the unique closed specialization of β from $\widetilde{O}$, then there is an open definable subset V of O such that $\beta \in \widetilde{V}$ and the closed specialization of β in $\widetilde{O}$ is not in $\widetilde{V}$, then replace O by V and proceed by induction on the number of proper specializations of β.) For this U we have that β is closed in $\widetilde{U}$. Thus since U is definably normal ([11] Chapter VI (3.5)), by [15] Theorem 2.13, there is an open definable subset W such that $W \subseteq \overline{W} \subseteq U$ and $\beta \in \widetilde{W}$. But then $\widetilde{(f_{|f^{-1}(\overline{W})})^{-1}(\beta)} = \widetilde{f^{-1}(\beta)}$. Now since $f^{-1}(\overline{W})$ is definably compact and $\widetilde{(f_{|f^{-1}(\overline{W})})^{-1}(\beta)}$ is a closed type definable subset of $\widetilde{f^{-1}(\overline{W})}$, arguing as in the proof of [2] Theorem 9.2 we conclude that

$$H^*(\widetilde{(f_{|f^{-1}(\overline{W})})^{-1}(\beta)}(S);F) \simeq H^*(\widetilde{(f_{|f^{-1}(\overline{W})})^{-1}(\beta)};F)$$

and so

$$H^*(\widetilde{f^{-1}(\beta)}(S);F) \simeq H^*(\widetilde{f^{-1}(\beta)};F).$$

□

Our next theorem holds with no restrictions on the definable spaces and sheaves. A continuous definable map $f : X \longrightarrow Y$ between definable spaces is *definably closed* if it maps every closed definable subset of X into a closed definable subset of Y.

Theorem 2.7 (Projection formula) *Let $f : X \longrightarrow Y$ be a continuous definable map between definable spaces which is definably closed. Let k be a commutative ring and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of k-modules on X and Y respectively with respect to the o-minimal site. Then there is a natural isomorphism*

$$Rf_*\mathcal{F} \otimes_k \mathcal{G} \simeq Rf_*(\mathcal{F} \otimes_k f^*\mathcal{G}).$$

Proof. This result follows from its version in the category of o-minimal spectra of definable spaces. In this context our result is similar to [21] Proposition 2.6.6 which uses Lemma 2.5.12 and Proposition 2.5.13 from [21]. Thus we need to verify the later results in our context.

Claim 2.8 *Let Z be the o-minimal spectra of a definable space, k a commutative ring and M a flat k-module. Let $\mathcal{F}$ be a sheaf of k-modules on Z. Then there is a natural isomorphism*

$$\Gamma(Z; \mathcal{F}) \otimes_k M \simeq \Gamma(Z; \mathcal{F} \otimes_k M).$$

In particular, if $\mathcal{F}$ is soft, then $\mathcal{F} \otimes_k M$ is soft.

For this we can follow the proof of Lemma 2.5.12 in [21].

Claim 2.9 *Let $f : Z \longrightarrow W$ be the o-minimal spectra of a continuous definable map which maps closed constructible subsets into closed subsets. Let k a commutative ring and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of k-modules on W and Z respectively. Then there is a natural morphism*

$$f_*\mathcal{G} \otimes_k \mathcal{F} \longrightarrow f_*(\mathcal{G} \otimes_k f^*\mathcal{F})$$

such that if $\mathcal{F}$ is k-flat, then it is an isomorphism.

For this claim we follow the proof of Proposition 2.5.13 in [21] by replacing the use of Proposition 2.5.2 and Lemma 2.5.12 there by Claim 2.5 (with $q = 0$ and the family of supports of closed constructible sets) and Claim 2.8 respectively.

With Claims 2.8 and 2.9 available we can finish the proof of the theorem as in [21] Proposition 2.6.6. First assume that $\mathcal{F}$ is a flat sheaf. By Claim 2.8, ones sees that the functor $-\otimes_k f^*\mathcal{F}$ sends soft sheaves into f_*-injective sheaves. Hence $Rf_*(-\otimes_k f^*\mathcal{F})$ is the derived functor of $f_*(-\otimes_k f^*\mathcal{F})$. Since $Rf_*(-)\otimes_k \mathcal{F}$ is the derived functor of $f_*(-)\otimes_k \mathcal{F}$, the result follows from Claim 2.9 in this case. For the general case we note that $\mathcal{F}$ as an object of the derived category of sheaves bounded from below is quasi-isomorphic to a complex bounded from below of flat sheaves. □

Let us give two important corollaries of the theorems above. The first one tells us that the cohomology of a definable space with coefficients in an arbitrary commutative group is known as soon as it is known over $\mathbb{Z}$. The second one tells us how to calculate the cohomology of a product of bounded, definably compact, definably normal definable spaces.

Corollary 2.10 (Universal coefficients formula) *Suppose that X is a definable space and let k be a commutative ring and M a k-module. Then there is natural isomorphism*

$$R\Gamma(X;M) \simeq R\Gamma(X;k)\otimes_k M$$

and

$$\begin{aligned} R\Gamma(X;M) &\simeq \textstyle\bigoplus_j H^j(X;M)[-j] \\ &\simeq \textstyle\bigoplus_j (H^j(X;\mathbb{Z})\otimes_{\mathbb{Z}} M \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H^{j+1}(X;\mathbb{Z}),M))[-j]. \end{aligned}$$

Proof. One has $M = a_X^* M \otimes_k k$ where $a_X : X \longrightarrow \{\mathrm{pt}\}$ is the constant map. By the projection formula (Theorem 2.7), we get

$$Ra_{X*}(a_X^*M\otimes_k k) \simeq Ra_{X*}k\otimes_k M.$$

Furthermore, since the homological dimension of the ring $\mathbb{Z}$ is one, we have for an object N of the bounded derived category of sheaves of $\mathbb{Z}$-modules

$$\begin{aligned} N &\simeq \oplus_j H^j(N)[-j], \\ N\otimes_{\mathbb{Z}} M &\simeq \textstyle\bigoplus_j (H^j(N)\otimes_{\mathbb{Z}} M \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H^{j+1}(N),M))[-j]. \end{aligned}$$

□

Corollary 2.11 (Künneth formula) *Let X and Y be bounded, definably compact, definably normal definable spaces. Let k be a commutative ring and let F and M be k-modules. Then*

$$R\Gamma(X\times Y;F\otimes_k M) \simeq R\Gamma(X;F)\otimes_k R\Gamma(Y;M).$$

Proof. Consider the diagram

$$\begin{array}{ccc} X & \stackrel{p_X}{\longleftarrow} & X \times Y \\ \downarrow^{a_X} & & \downarrow^{p_Y} \\ \mathrm{pt} & \stackrel{a_Y}{\longleftarrow} & Y. \end{array}$$

Then by the projection formula (Theorem 2.7) and the proper base change (Theorem 2.4) we have

$$\begin{array}{rcl} Ra_{X\times Y*}(F \otimes_k M) & \simeq & Ra_{Y*}Rp_{Y*}(p_X^*F \otimes_k p_Y^*M) \\ & \simeq & Ra_{Y*}((Rp_{Y*}p_X^*F) \otimes_k M)) \\ & \simeq & Ra_{Y*}(a_Y^*Ra_{X*}F \otimes_k M) \\ & \simeq & Ra_{X*}F \otimes_k Ra_{Y*}M. \end{array}$$

□

3 On torsion points

Here we apply the results of Section 2 to definably compact definable groups and prove Theorem 1.1 from the introduction.

Let G be a definably compact, definably connected, definable abelian group. In order to prove Theorem 1.1 we consider three case.

Case 1: $\mathcal{R}$ is an o-minimal expansion of a field. In this case it was proved in [18] using o-minimal singular cohomology that the o-minimal fundamental group of G is $\pi_1(G) \simeq \mathbb{Z}^{\dim G}$ and hence the subgroup of k-torsion points of G is $G[k] \simeq (\mathbb{Z}/k\mathbb{Z})^{\dim G}$.

Case 2: $\mathcal{R}$ is a linear o-minimal expansion of a group. By the work of Lovey's and Peterzil in [22], this means that an elementary extension of $\mathcal{R}$ is a reduct of an ordered vector space over an ordered division ring. So one may assume that $\mathcal{R}= (R, 0, 1, +, <, (\lambda)_{\lambda\in\Lambda})$ (an ordered vector space over an ordered division ring Λ). In this situation it was proved by Eleftheriou and Starchenko in [19] by a direct analysis of the structure of definable groups that the o-minimal fundamental group of G is $\pi_1(G) \simeq \mathbb{Z}^{\dim G}$ and the subgroup of k-torsion points of G is $G[k] \simeq (\mathbb{Z}/k\mathbb{Z})^{\dim G}$.

Case 3: $\mathcal{R}$ is a (non linear) semi-bounded o-minimal expansion of a group.

Lemma 3.1 *Assume that $\mathcal{R}$ is a semi-bounded o-minimal expansion of a group. Then every definably compact definable group is bounded as a definable set and as a definable space.*

Proof. Let G be a definably compact definable group with $G \subseteq R^l$. Suppose that G is not a bounded definable subset of R^l. Then by definable choice ([11] Chapter VI, Proposition 1.2) we have a definable map $\alpha : (a, +\infty) \longrightarrow G$, which by o-minimality we may assume to be injective and continuous with respect to the definable manifold structure on G given in [27]. Since G is definably compact, the limit $\lim_{t \longrightarrow +\infty} \alpha$ exists in G. Let b be this limit and let $g \in G$ be a generic (over the set of parameters over which G is defined). Then replacing α by $(a, +\infty) \longrightarrow G : t \mapsto gb^{-1}\alpha(t)$ (where the products and inverse are taken in G), we may assume that b is generic. In this situation, by [27], a fundamental system of definable neighborhoods of b in G is given by the intersection of a fundamental system of definable neighborhoods of b in R^l with G. In particular, by continuity and o-minimality, we can assume that the image of α is contained in a bounded open box containing b. As the image $\mathrm{Im}\alpha$ of α is a bounded one-dimensional definable subset of G, by applying cell decomposition to $\mathrm{Im}\alpha$, we conclude that there exists a definable bijection between an unbounded interval and a bounded interval, contradicting the fact that $\mathcal{R}$ is semi-bounded, i.e., it has no poles ([12]).

Finally, since G is bounded as a definable set, by the construction of the definable manifold structure of G given in [27], G is also bounded as a definable space. □

By Lemma 10.4 in [3] Hausdorff definable manifolds in o-minimal expansions of fields are definably normal definable spaces. Since the same proof holds in o-minimal expansions of groups (it only uses the existence of a norm in R^l) and definable groups are Hausdorff, we have:

Lemma 3.2 *Let $\mathcal{R}$ be an arbitrary o-minimal expansion of an ordered group. If G is a definably compact definable group, then G is a definably normal definable space.*

We will also require the following general result:

Theorem 3.3 (Hurewicz theorem) *Let $\mathcal{R}$ be an arbitrary o-minimal expansion of an ordered group. If G is a definably connected locally definable group and Z a countable abelian group, then we have a homomorphism*

$$\mathrm{Hom}(\pi_1(G), Z) \longrightarrow \check{H}^1(G; Z),$$

which does not send any sujective elements of $\mathrm{Hom}(\pi_1(G), Z)$ *to zero.*

Proof. By Theorem 1.4 in [14] we have an isomorphism between the fundamental group $\pi_1(G)$ of G defined using definable paths and the fundamental group $\pi(G)$ of G defined using locally definable covering homomorphisms.

The proof of the Hurewicz theorem for definable groups in arbitrary o-minimal structures presented in [16] shows with the same arguments a Hurewicz theorem for $\pi(G)$, namely, for any definably connected locally definable group G in an arbitrary o-minimal structure, and any (abstract) countable abelian group Z, there is a homomorphism

$$\mathrm{Hom}_{\mathrm{fact}}(\pi(G), Z) \longrightarrow \check{H}^1(G; Z)$$

with the extra important property that it does not send any surjective elements of $\mathrm{Hom}_{\mathrm{fact}}(\pi(G), Z)$ to zero. Here $\mathrm{Hom}_{\mathrm{fact}}(\pi(G), Z)$ is the subgroup of $\mathrm{Hom}(\pi(G), Z)$ of all homomorphisms $h : \pi(G) \longrightarrow Z$ such that there exists $p : H \longrightarrow G \in \mathrm{Cov}^0(G)$ and a homomorphism $\overline{h} : \mathrm{Ker}p \longrightarrow Z$ with $h = \overline{h} \circ \theta_H$ where $\theta_H : \pi(G) \longrightarrow \mathrm{Ker}p$ is the natural projection. The group $\check{H}^1(G; Z)$ is the first Čech cohomology group of G with coefficients in Z. O-minimal Čech cohomology for the definable category was defined in [17] where the authors also prove the Eilenberg-Steenrod axioms for this theory. The construction of the Čech cohomology groups easily generalizes to locally definable groups by considering countable covers by open definable subsets.

By Theorem 1.4 in [14] again we have $\mathrm{Hom}_{\mathrm{fact}}(\pi(G), Z) = \mathrm{Hom}(\pi(G), Z)$. Indeed, if $h : \pi(G) \longrightarrow Z$ is in $\mathrm{Hom}(\pi(G), Z)$, then we have a canonical $p : \widetilde{G}/\mathrm{Ker}h \longrightarrow G \in \mathrm{Cov}^0(G)$ such that the universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ is the composition of the quotient locally definable covering homomorphism $\widetilde{G} \longrightarrow \widetilde{G}/\mathrm{Ker}h$ and $p : \widetilde{G}/\mathrm{Ker}h \longrightarrow G$. Since $\mathrm{Ker}p \simeq \pi(G)/\mathrm{Ker}h$ we find a homomorphism $\overline{h} : \mathrm{Ker}p \longrightarrow Z$ with the right properties. □

We can now give a partial computation of the cohomology groups of definably compact definable groups:

Theorem 3.4 *Assume that $\mathcal{R}$ is a semi-bounded o-minimal expansion of a group. Let G be a definably connected, definably compact definable group and let $m : G \times G \longrightarrow G$ be the multiplication in G. Then $H^*(G; \mathbb{Q})$ can be equipped with a structure of a Hopf-algebra over $\mathbb{Q}$, with comultiplication $m^* : H^*(G; \mathbb{Q}) \longrightarrow H^*(G; \mathbb{Q}) \otimes H^*(G; \mathbb{Q})$. Furthermore we have an isomorphism*

$$H^*(G; \mathbb{Q}) \simeq \wedge[\omega_1, \ldots, \omega_r]_{\mathbb{Q}}$$

of Hopf-algebra over $\mathbb{Q}$, with the ω_i's of odd degree and primitive, that is, $m^*(\omega_i) = \omega_i \otimes 1 + 1 \otimes \omega_i$, for each $i = 1, \ldots, r$.

Proof. By Lemmas 3.1 and 3.2, G is a bounded, definably normal definable space, hence we can use the Künneth formula (Corollary 2.11) as in Theorem 3.4 and Corollaries 3.5 and 3.6 in [18] to finish the proof of the theorem. $\square$

Proof of Theorem 1.1 in Case 3. By Corollary 1.5 in [14] we have $\pi_1(G) \simeq \mathbb{Z}^s$ for some s. By Theorems 3.3 and 3.4 and the fact that $\check{H}^*(G;\mathbb{Q}) \simeq H^*(G;\mathbb{Q})$ (Proposition 4.1 in [15]), we have $s \leq r$. On the other hand, $r \leq \deg\omega_1 + \cdots + \deg\omega_r = q$ and $0 \neq \omega_1 \wedge \cdots \wedge \omega_r \in H^q(G;\mathbb{Q})$. Since $H^p(G;\mathbb{Q}) = 0$ for $p > \dim G$ (by Theorem 2.2 or [15] Proposition 4.2) we have $q \leq \dim G$ and so $s \leq \dim G$ as required.

Combining the above proof and the main result from [19] we obtain:

Corollary 3.5 *Assume that $\mathcal{R}$ is a linear o-minimal expansion of a group. Let G be a definably connected, definably compact definable group. Then we have an isomorphism*

$$H^*(G;\mathbb{Q}) \simeq \wedge[\omega_1, \ldots, \omega_{\dim G}]_{\mathbb{Q}}$$

of Hopf-algebra over $\mathbb{Q}$, with the ω_i's of degree one.

4 Appendix: the general case

Here $\mathcal{M}$ is an arbitrary o-minimal structure with definable Skolem functions and definable means definable in $\mathcal{M}$. As we explained in the introduction we now present the proof of Theorem 1.2. Since we will need a result on Hopf-algebras over $\mathbb{Z}/p\mathbb{Z}$ which we could not find explicitly stated in the literature we also include here a subsection on this.

4.1 The general case: Theorem 1.2

First we explain how to obtain a general version of the Künneth formula (Corollary 2.11) for definably normal definable spaces. For this it is enough to prove a proper base change theorem (Theorem 2.4) for definably normal definable spaces.

Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be continuous definable maps between definably compact, definably normal definable spaces. Form the cartesian square

$$\begin{array}{ccc} X & \stackrel{g'}{\longleftarrow} & X \times_Y Z \\ \downarrow^{f} & & \downarrow^{f'} \\ Y & \stackrel{g}{\longleftarrow} & Z. \end{array}$$

Let $\mathcal{F}$ be a sheaf on X with respect to the o-minimal site. Then there is a canonical base change homomorphism

$$\alpha : g^* R^q f_* \mathcal{F} \longrightarrow R^q f'_*(g'^* \mathcal{F})$$

for every $q \geq 0$.

Theorem 4.1 (General proper base change) *Suppose that X has a uniformly definable system of fundamental open definable neighborhoods. If F is an abelian group, then the base change homomorphism*

$$\alpha : g^* R^q f_* F \longrightarrow R^q f'_*(g'^* F)$$

is an isomorphism for every $q \geq 0$.

Proof. Looking at the proof of Theorem 2.4, we see that the only place where we used the assumption that we are in an o-minimal expansion of a group and we are working with bounded, definably compact, definably normal definable spaces was in the proof of Claim 2.6. In that setting, by the proof of Lemma 3.1 in [1] and our assumptions on X and $\mathcal{M}$, we have a homeomorphism $\widetilde{f^{-1}}(\beta)(S) \simeq \widetilde{f^{-1}}(\beta)$ of topological spaces. Thus this claim also holds in $\mathcal{M}$ under the assumptions we are assuming. $\square$

As in Corollary 2.11 we obtain from Theorem 4.1:

Corollary 4.2 (General Künneth formula) *Let X and Y be definably compact, definably normal definable spaces with uniformly definable systems of fundamental open definable neighborhoods. Let k be a commutative ring and let F and M be k-modules. Then*

$$R\Gamma(X \times Y; F \otimes_k M) \simeq R\Gamma(X; F) \otimes_k R\Gamma(Y; M).$$

Next we need a generalization of Theorem 3.4. But first we need to show that definably compact definable groups are definably normal spaces. For that we will require the following remark:

Remark 4.3 Let G be a definably connected, definable group. Suppose that $n = \dim G$. By the proof of [13] Lemma 2.3, G has a definable open neighborhood O of the identity e_G of G and a definable homeomorphism $\phi : O \longrightarrow M^n$ such that $\phi(O) = \Pi_{i=1}^n I_i$ where each I_i is an interval of M of the form $(-_i e_i, e_i)$ inside an ordered definable group $(J_i, 0_i, +_i, -_i, <_i)$. So G has a uniformly definable system of fundamental open definable neighborhoods.

The following proposition will also be required.

Proposition 4.4 *Let G be a definably connected, definable group. Then G is definably locally compact, i.e. for every definably compact definable subset K of G and every open definable neighborhood U of K in G, there exists a definably compact neighborhood V of K in U.*

Proof. As above, let $n = \dim G$ and let (O, ϕ) be the definable open chart of the identity e_G of G given by the proof of [13] Lemma 2.3.

Let K be a definably compact definable subset of G and U an open definable neighborhood of K in G. We have a uniformly definable family $\{a^{-1}U \cap O : a \in K\}$ of definable open neighborhood of e_G in O. By [13] Lemma 2.3, O has definable choice, so there is a uniformly definable family $\{V_a : a \in K\}$ of definably compact definable neighborhoods of e_G in O of the form $V_a = \phi^{-1}(\Pi_{i=1}^n [-_i t_i^a, t_i^a]) \subseteq a^{-1}U \cap O$. Now take $V = \cup\{V_a : a \in K\}$. Then V is a closed definable neighborhood of K in U. It remains to show that it is definably compact.

Let $\alpha : (b, c) \subseteq M \longrightarrow V$ be a continuous definable map. Since G has definable choice ([13] Theorem 7.2), so thus V. Hence, there is a continuous definable map $\beta : (b, c) \subseteq M \longrightarrow K$ such that for all $t \in (b, c)$ we have $\alpha(t) \in V_{\beta(t)}$. Since K is definably compact, there is $d \in K$ such that $\lim_{t \longrightarrow c^-} \beta(t) = d$. But then there is $b < e < c$ such that $\alpha(e, c) \subseteq V_d$. As V_d is definably compact, $\lim_{t \longrightarrow c^-} \alpha(t) \in V_d \subseteq V$ as required. □

Proposition 4.5 *Let G be a definably connected, definably compact definable group. Then G is a definably normal space.*

Proof. As above, let $n = \dim G$ and let (O, ϕ) be the definable open chart of the identity e_G of G given by the proof of [13] Lemma 2.3.

First notice that G is definably regular i..e. if C is a closed definable subset of G and a is a point of G not in C, then there exist open, disjoint definable neighborhoods U and V of a and C respectively in G. Indeed, if $C \cap aO = \emptyset$, let $W = a\phi^{-1}(\Pi_{i=1}^n [-_i t_i, t_i]) \subseteq aO$ and take $U = a\phi^{-1}(\Pi_{i=1}^n (-_i t_i, t_i))$ and $V = G \setminus W$. If $C \cap aO \neq \emptyset$, then since O is definably regular, there exists $S = a\phi^{-1}(\Pi_{i=1}^n[-_i s_i, s_i]) \subseteq aO$ such that $S \subseteq aO \setminus C$ and take $U = a\phi^{-1}(\Pi_{i=1}^n(-_i t_i, t_i))$ with $t_i <_i s_i$ and $V = G \setminus S$.

Now let (K, C) be a pair of closed, disjoint definable subsets of G. We prove the result by induction on $\dim K$. By the previous paragraph the result holds for $\dim K = 0$. Suppose it holds for pairs of closed , disjoint definable subsets (L, C) of G with $\dim L < \dim K$. Since $\phi(O)$ has definable choice, there exist a definable map $g = (g_1, \dots, g_n) : K \longrightarrow \phi(O) : a \mapsto (t_1^a, \dots t_n^a)$ such that $U_a = a\phi^{-1}(\Pi_{i=1}^n(-_i t_i^a, t_i^a))$ is an open definable neighborhood of a in G disjoint from C. The definable subset of K on which g is not continuous, is a definable set of dimension strictly less than $\dim K$. Let L be the closure of this set in K. Then $\dim L < \dim K$ and (L, C) is a pair of disjoint, closed definable subsets of G. So there exist disjoint, open definable neighborhoods (in G) U_L and V_L of L and C respectively.

By Proposition 4.4, there is a definably compact definable neighborhood W of L in U_L. Take L' to be the intersection of K with the complement in G of the interior of W (in G). Then L' is definably compact and $g_{|L'} : L' \longrightarrow \phi(O)$ is continuous. Fix $i \in \{1, \dots, n\}$. Then for all $a \in L'$, $0_i <_i t_i^a = g_i(a)$. We show that there is $0_i <_i d_i$ such that $d_i \leq t_i^a = g_i(a)$ for all $a \in L'$. Suppose not. Then for all $0_i <_i s$ there exists $a \in L'$ such that $0_i <_i g_i(a) <_i s$. Since G has definable choice ([13] Theorem 7.2), so thus L'. Hence, there is a definable map $\alpha_i : (0_i, c_i) \subseteq I_i \longrightarrow L'$ such that for all $0_i <_i t <_i c_i$ we have $0_i <_i g_i(\alpha_i(t)) <_i t$. By o-minimality we may assume that α_i is continuous. Since L' is definably compact there is $d \in L'$ such that $\lim_{t \longrightarrow 0_i^{+_i}} \alpha_i(t) = d$. So $g_i(d) = g_i(\lim_{t \longrightarrow 0_i^{+_i}} \alpha_i(t)) = \lim_{t \longrightarrow 0_i^{+_i}} g_i(\alpha_i(t)) = 0$ which is a contradiction.

By construction, for all $a \in L'$, $a\phi^{-1}(\Pi_{i=1}^n(-_i d_i, d_i))$ is an open definable neighborhood of a in G disjoint from C. Consider now the definable sets

$$U_{L'} = \cup\{a\phi^{-1}(\Pi_{i=1}^n(-_i \frac{d_i}{2}, \frac{d_i}{2})) : a \in L'\}$$

and

$$V_{L'} = G \setminus \cup\{a\phi^{-1}(\Pi_{i=1}^n[-_i \frac{3d_i}{2}, \frac{3d_i}{2}]) : a \in L'\}.$$

Then $U_{L'}$ and $V_{L'}$ are disjoint, open definable neighborhoods (in G) of L' and C respectively. Finally take $U = U_L \cup U_{L'}$ and $V = V_L \cap V_{L'}$. Then U and V are disjoint, open definable neighborhoods (in G) of K and C respectively. $\square$

We are ready to prove the general version of Theorem 3.4 giving a description of the o-minimal cohomology of definably connected, definably compact, definable groups.

Theorem 4.6 *Let G be a definably connected, definably compact definable group and let k be field. Then $H^*(G;k)$ can be equipped with a structure of a Hopf-algebra over k with comultiplication induced by the multiplication in G.*

Proof. By Remark 4.3 and Proposition 4.5, G is a definably normal definable space with a uniformly definable system of fundamental open definable neighborhoods. Therefore we can use the general Künneth formula (Corollary 4.2) as in Theorem 3.4 and Corollaries 3.5 and 3.6 in [18] to finish the proof of the theorem. □

Proof of Theorem 1.2. Let G be a definably connected, definably compact definable abelian group. First we recall some results from [16]. The classifying group $\eta(G)$ of G is a profinite abelian group ([16] Theorem 2.2) and so

$$\eta(G) = \Pi_{p \text{ prime}}\Pi_{\kappa(p)}\mathbb{Z}_p$$

where for each prime p, $\mathbb{Z}_p$ is the group of p-adic integers and $\kappa(p)$ is some cardinal number. By the general Hurewicz theorem (Theorem 1.1 in [16]) we have a homomorphism

$$\mathrm{Hom}_{\mathrm{cont}}(\eta(G), \mathbb{Z}/p\mathbb{Z}) \longrightarrow \check{H}^1(G; \mathbb{Z}/p\mathbb{Z}),$$

where $\mathrm{Hom}_{\mathrm{cont}}(\eta(G), \mathbb{Z}/p\mathbb{Z})$ is the group of all continuous homomorphisms from the profinite group $\eta(G)$ into $\mathbb{Z}/p\mathbb{Z}$. Further, this homomorphism does not send any sujective elements of $\mathrm{Hom}_{\mathrm{cont}}(\eta(G), \mathbb{Z}/p\mathbb{Z})$ to zero.

Now observe also that, since G is definably normal (Proposition 4.5), we have $\check{H}^*(G;\mathbb{Z}/p\mathbb{Z}) \simeq H^*(G;\mathbb{Z}/p\mathbb{Z})$ (Proposition 4.1 in [15]) and also $H^p(G;\mathbb{Z}/p\mathbb{Z}) = 0$ for $p > \mathrm{dim} G$ (Proposition 4.2 in [15]).

By Theorem 4.8 on Hopf-algebras over $\mathbb{Z}/p\mathbb{Z}$ (and the last observation on the previous paragraph), if $p > \dim G + 1$, then

$$H^*(G;\mathbb{Z}/p\mathbb{Z}) \simeq \bigwedge[\omega_1, \ldots, \omega_t]_{\mathbb{Z}/p\mathbb{Z}}$$

for some $t \leq \dim G$ with $\sum_{i=1}^{t} \mathrm{deg}\omega_i \leq \dim G$ and the ω_i's of odd degree and primitive, that is, $\mu(\omega_i) = \omega_i \otimes 1 + 1 \otimes \omega_i$, for each $i = 1, \ldots, t$. Thus, if $p > \dim G + 1$, then since by the general Hurewicz theorem each generator of $\mathrm{Hom}_{\mathrm{cont}}(\eta(G), \mathbb{Z}/p\mathbb{Z})$ determines a generator of $H^*(G;\mathbb{Z}/p\mathbb{Z})$ of degree one, it follows that $\kappa(p) \leq t \leq \dim G$ as required.

4.2 Graded Hopf algebras over finite fields

Let $H = \sum_{k \geq 0} H_k$ be a graded, skew-commutative, associative R-algebra with unity element where R is a commutative ring with unit. We say that H is *connected* if $H_0 \simeq R$.

Definition 4.7 We will call a connected, graded, skew-commutative, associative R-algebra $H = \sum_{k\geq 0} H_k$ with unity element a *graded quasi Hopf R-algebra* (of finite type) if each H_k is an R-module (resp., a finite dimensional R-module) and there is a degree preserving R-algebra homomorphism $\psi : H \longrightarrow H \otimes_R H$ called *co-multiplication or diagonal* such that: if e is a generator of $H_0 \simeq R$ as an R-algebra of dimension one, then the map $\epsilon : H \longrightarrow R$, defined by $\epsilon(e) = 1$ and $\epsilon(x) = 0$ for all $x \in H_k$ with $k \geq 1$ is a *co-unit* i.e., for all $x \in H$,

$$(\epsilon \otimes_R 1)\psi(x) = 1 \otimes_R x \quad \text{and} \quad (1 \otimes_R \epsilon)\psi(x) = x \otimes_R 1.$$

A graded quasi Hopf R-algebra H is called a *graded Hopf R-algebra* if ψ is associative i.e.,

$$(\psi \otimes_R 1)\psi = (1 \otimes_R \psi)\psi;$$

and we say that ψ is commutative if we have $T \circ \psi = \psi$ where $T(x \otimes_R y) = (-1)^{\deg(x)\deg(y)} y \otimes_R x$.

The main result of these notes is the following:

Theorem 4.8 *Let $p \geq 2$ be a prime and let H be a graded Hopf $\mathbb{Z}/p\mathbb{Z}$-algebra. Then there is a free, skew-commutative, graded Hopf $\mathbb{Z}/p\mathbb{Z}$-algebra*

$$A = \mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_k, \ldots] \otimes_{\mathbb{Z}/p\mathbb{Z}} \bigwedge[y_1, \ldots, y_l, \ldots]_{\mathbb{Z}/p\mathbb{Z}}$$

and there is a homomorphism $h : A \longrightarrow H$ of connected, graded, skew-commutative, associative split $\mathbb{Z}/p\mathbb{Z}$-algebras with unity element such that h is surjective and $h_j : A_j \longrightarrow H_j$ is an isomorphism for all $j \leq p-2$.

In the case of graded Hopf algebras this is a weak analogue of the Hopf-Leray-Borel theorem for graded Hopf algebras over $\mathbb{Z}/p\mathbb{Z}$ which are not necessarily of finite type (for details see [24] Chapter VII, Corollary 1.4 and also [5] and [23]):

Theorem 4.9 (Hopf-Leray-Borel) *Let H be a graded quasi Hopf algebra of finite type over a perfect field K_p of characteristic p. Then we have the following ring isomorphisms:*
(1) For $p = 0$; $H \simeq (\bigwedge_\alpha [x_\alpha]_{K_0}) \otimes (\bigotimes_\beta K_0[x_\beta])$, where $\deg x_\alpha$ is odd and $\deg x_\beta$ is even.
(2) For $p = 2$; $H \simeq (\bigotimes_\alpha K_2[x_\alpha]/(x_\alpha^{h_\alpha})) \otimes (\bigotimes_\beta K_2[x_\beta])$, where h_α is a power of 2.
(3) For $p \neq 0, 2$; $H \simeq (\bigwedge_\alpha [x_\alpha]_{K_p}) \otimes (\bigotimes_\beta K_p[x_\beta]) \otimes (\bigotimes_\gamma K_p[x_\gamma]/(x_\gamma^{h_\gamma}))$, where $\deg x_\alpha$ is odd, $\deg x_\beta$ and $\deg x_\gamma$ are even, and h_γ is a power of p.

Here, if $\dim H < \infty$, then there is no term of $K_p[x_\beta]$.

In [10] Chapter VII, Proposition 10.16, it is proved that in the characteristic zero case, Theorem 4.9 holds for graded quasi Hopf algebras which are not necessarily of finite type. This is the original Hopf-Leray theorem in characteristic zero.

To prove Theorem 4.8 we will require first some generalities about connected, graded, skew-commutative associative R-algebra with unity element. The argument will follow the treatment of the characteristic zero case presented in [10] Chapter VII, Section 10.

Definition 4.10 Let $A = \sum_{k\geq 0} A_k$ be a connected, graded, skew-commutative associative R-algebra with unity element. Let $\mu : A \otimes_R A \longrightarrow A$ be the multiplication. We will often write $a_1 a_2$ for $\mu(a_1, a_2)$.

Define the graded R-submodules $D^n A$ of A where $n = 0, 1, \ldots$, as follows: $D^0 A = A$; $(D^1 A)_j = A_j$ for $j > 0$ and $(D^1 A)_j = 0$ for $j \leq 0$; and $D^{n+1} A = \mathrm{Im}(\mu : D^n A \otimes D^1 A \longrightarrow A)$. Clearly $D^{n+1} A \subseteq D^n A$ and $(D^n A)_j = 0$ for all $j < n$.

Define the R-modules $\Theta^n A = D^n A / D^{n+1} A$. We say that A is a *split R-algebra* if $D^{n+1} A$ is a direct summand of $D^n A$ for all n. For example, if R is a field, then A is always a split R-algebra.

Remark 4.11 Note that, by [10] Chapter VII, Proposition 10.4, D^n and Θ^n may be viewed, in a canonical way (by restriction and quotient respectively), as functors of connected, graded, skew-commutative, associative R-algebra with unity element.

Another observation that we should make is that the R-module $\Theta^1 A = D^1 A / D^2 A$ will play a special role in what follows. Indeed, if $M \subseteq D^1 A$ is an R-submodule which maps epimorphically onto the quotient $\Theta^1 A$, then M generates the R-algebra A. In particular, if $h : B \longrightarrow A$ is a homomorphism of connected, graded R-algebras such that $\Theta^1 h : \Theta^1 B \longrightarrow \Theta^1 A$ is surjective, then h is surjective. For details on this see [10] Chapter VII, Lemma 10.5 and Corollary 10.6.

By [10] Chapter VII, (10.3) and Proposition 10.7, we have the following easy result.

Remark 4.12 Let A and B be connected, graded, skew-commutative, associative split R-algebra with unity element. Then $A \otimes_R B$ is also a connected, graded, skew-commutative, associative split R-algebra with unity element. Moreover, we have

$$D^n A \simeq \Theta^n A \oplus D^{n+1} A \simeq \bigoplus_{k\geq n} \Theta^k A,$$

$$D^n(A \otimes_R B) = \sum_{0 \le i \le n} D^i A \otimes_R D^{n-i} B$$

and

$$\Theta^n(A \otimes_R B) = \sum_{0 \le i \le n} \Theta^i A \otimes_R \Theta^{n-i} B.$$

Below, we will use $\pi^i_A : \Theta^n(A \otimes_R A) \longrightarrow \Theta^i A \otimes_R \Theta^{n-i} A$ to denote the natural projection and we will use $\iota^i_A : \Theta^i A \otimes_R \Theta^{n-i} A \longrightarrow \Theta^n(A \otimes_R A)$ to denote the natural inclusion.

The proof of Lemma 4.13 below is the same as [10] Chapter VII, Lemma 10.11.

Lemma 4.13 *Let $h : A \longrightarrow H$ be a homomorphism of connected, graded, skew-commutative, associative split R-algebras with unity element. Suppose that H is a graded Hopf R-algebra. Then the multiplication $\mu : A \otimes_R A \longrightarrow A$ of A and the comultiplication $\psi : H \longrightarrow H \otimes_R H$ of H are R-algebra homomorphisms. Moreover, if $\Theta^i h : \Theta^i A \longrightarrow \Theta^i H$ and $\Theta^{n-i} h : \Theta^{n-i} A \longrightarrow \Theta^{n-i} H$ are isomorphisms and*

$$\theta^n_i h = (\Theta^n \mu) \circ \iota^i_A \circ (\Theta^i h \otimes_R \Theta^{n-i} h)^{-1} \circ \pi^i_H \circ (\Theta^n \psi) \circ (\Theta^n h),$$

then $\theta^n_i h : \Theta^n A \longrightarrow \Theta^n A$ agrees with the map $\frac{n!}{i!(n-i)!} \cdot 1 : \Theta^n A \longrightarrow \Theta^n A$.

Proof. It is enough to show that $\theta^n_i h : \Theta^n A \longrightarrow \Theta^n A$ agrees with the map $\frac{n!}{i!(n-i)!} \cdot 1 : \Theta^n A \longrightarrow \Theta^n A$ on the generators $a_1 a_2 \cdots a_n \in \Theta^n A$ where $a_x \in \Theta^1 A \subseteq D^1 A$. Let $b_x = h(a_x)$. Then we have $\psi(b_x) = b_x \otimes_R 1 + 1 \otimes_R b_x + r_x$ with $r_x \in D^2(H \otimes_R H)$, hence

$$\psi \circ h(a_1 a_2 \cdots a_n) = \psi(b_1 b_2 \cdots b_n) = \Pi_x(b_x \otimes_R 1 + 1 \otimes_R b_x) + r$$

with $r \in D^{n+1}(H \otimes_R H)$.

The image of $\Pi_x(b_x \otimes_R 1 + 1 \otimes_R b_x)$ by the projection $\pi^i_H : \Theta^n(H \otimes_R H) \longrightarrow \Theta^i H \otimes_R \Theta^{n-i} H$ is

$$b = \sum \sigma_{x_1,\ldots,x_i} b_{x_1} \cdots b_{x_i} \otimes_R b_{y_1} \cdots b_{y_{n-i}}$$

where $\sigma_{x_1,\ldots,x_i} \in \{-\,,+\}$, the sum extends over all i-tuples $1 \le x_1 < x_2 < \cdots < x_i \le n$, $y_1 < y_2 < \cdots < y_{n-i}$ is the complement of $\{x_1, x_2, \ldots, x_i\}$ in $\{1, \ldots, n\}$ and the signs $\sigma_{x_1,\ldots,x_i}$ are caused by the commutation law $(1 \otimes_R c_y)(c_x \otimes_R 1) = (-1)^{\mathrm{deg} c_y \mathrm{deg} c_x} c_x \otimes_R c_y$.

Consider the corresponding expression

$$a = \sum \sigma_{x_1,\ldots,x_i} a_{x_1} \cdots a_{x_i} \otimes_R a_{y_1} \cdots a_{y_{n-i}}$$

in $\Theta^i A \otimes_R \Theta^{n-i} A$. Then

$$(\Theta^i h \otimes_R \Theta^{n-i} h)(a) = b = \pi_H^i \circ \psi \circ h(a_1 a_2 \cdots a_n).$$

On the other hand, if we apply multiplication $\mu : A \otimes_R A \longrightarrow A$ to a, each summand goes into $a_1 a_2 \cdots a_n$ (the signs disappear when we reverse the permutation), and the number of summands is $\frac{n!}{i!(n-i)!}$. □

The next lemma is a modification of [10] Chapter VII, Proposition 10.14. The statement is different but the proof is essentially the same.

Lemma 4.14 *Let $p \geq 2$ be an integer. Let $h : A \longrightarrow H$ be a homomorphism of connected, graded, skew-commutative, associative split R-algebras with unity element. Suppose that H is a graded Hopf R-algebra and $D^1 A$ (as an abelian group) is k-torsion free for all $k < p$. If $\Theta^1 h : \Theta^1 A \longrightarrow \Theta^1 H$ is an isomorphism, then h is surjective and $h_j : A_j \longrightarrow H_j$ is an isomorphism for all $j \leq p - 2$.*

Proof. By Remark 4.11, h is surjective. Since h is surjective, it follows easily from the definition that $D^n h : D^n A \longrightarrow D^n H$ is surjective for all n. But then, by the first equation on Remark 4.12, it follows that $\Theta^n h : \Theta^n A \longrightarrow \Theta^n H$ is also surjective for all n. We will show that $\Theta^n h : \Theta^n A \longrightarrow \Theta^n H$ is injective for all $1 \leq n < p$. Let $1 < n < p$. By induction assume that $\Theta^{n-1} h : \Theta^{n-1} A \longrightarrow \Theta^{n-1} H$ is an isomorphism. Then, by Lemma 4.13, with $i = n-1$, we see that $\theta^n_{n-1} h : \Theta^n A \longrightarrow \Theta^n A$ agrees with $n \cdot 1 : \Theta^n A \longrightarrow \Theta^n A$. By the first equation on Remark 4.12 and the assumption on $D^1 A$, we see that $\Theta^n A$ is n-torsion free. Hence, $\theta^n_{n-1} h$ is injective. In particular, the first factor $\Theta^n h$ in the composition defining the homomorphism $\theta^n_{n-1} h$ is injective.

By decreasing induction on $n < p$ we now show that $(D^n h)_j : (D^n A)_j \longrightarrow (D^n H)_j$ is an isomorphism for all $j \leq p - 2$. For $n = 0$ we get the lemma. Let $n = p - 1$. Since $j < n$, we have $(D^n A)_j = (D^n H)_j = 0$. On the other hand, by definition, we have the exact sequence $0 \to D^n A \to D^{n-1} A \to \Theta^{n-1} A \to 0$. In particular, for each $j \leq p - 2$, we have the exact sequence $0 \to (D^n A)_j \to (D^{n-1} A)_j \to (\Theta^{n-1} A)_j \to 0$. Now the inductive step follows from the exact sequence $0 \to (D^n A)_j \to (D^{n-1} A)_j \to (\Theta^{n-1} A)_j \to 0$ and the five lemma. □

Proof of Theorem 4.8: Since $\mathbb{Z}/p\mathbb{Z}$ is a field, the $\mathbb{Z}/p\mathbb{Z}$-module $\Theta^1 H = D^1 H / D^2 H$ has a basis over $\mathbb{Z}/p\mathbb{Z}$. Let $M \subseteq D^1 H \subseteq H$ be the lifting of a basis of $\Theta^1 H$ over $\mathbb{Z}/p\mathbb{Z}$. Let $A = \mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_k, \ldots] \otimes_{\mathbb{Z}/p\mathbb{Z}} \bigwedge[y_1, \ldots, y_l, \ldots]_{\mathbb{Z}/p\mathbb{Z}}$ where each x_i (resp., y_j) is in M and the degree of x_i (resp., y_j) in A is the same as its degree in H. Let $h : A \longrightarrow H$ be the unique homomorphism of connected, graded, skew-commutative, associative split $\mathbb{Z}/p\mathbb{Z}$-algebras with unity element extending the inclusion $M \longrightarrow H$.

The result will follow from Lemma 4.14. So we need to verify the conditions of the lemma. Clearly, the set $\{x + D^2A : x \in M\}$ is a basis for the $\mathbb{Z}/p\mathbb{Z}$-module $\Theta^1 A = D^1A/D^2A$. Hence, $\Theta^1 h : \Theta^1 A \longrightarrow \Theta^1 H$ is an isomorphism.

We will show that D^1A is k-torsion free for every $k < p$. Let $0 < k < p$ and let $a \in D^1A$ be an element such that $ka = 0$. Since the elements of the form $x_{i_1} \cdot \cdots \cdot x_{i_l} \otimes_{\mathbb{Z}/p\mathbb{Z}} y_{j_1} \wedge \cdots \wedge y_{j_n}$ with $j_1 < \cdots < j_n$ are a basis for A over $\mathbb{Z}/p\mathbb{Z}$, we have that a is a finite sum $\sum_{i_1,\dots,i_l,\, j_1<\cdots<j_n} r^{j_1,\dots,j_n}_{i_1,\dots,i_l} x_{i_1} \cdot \cdots \cdot x_{i_l} \otimes_{\mathbb{Z}/p\mathbb{Z}} y_{j_1} \wedge \cdots \wedge y_{j_n}$. Therefore, $ka = \sum_{i_1,\dots,i_l,\, j_1<\cdots<j_n} kr^{j_1,\dots,j_n}_{i_1,\dots,i_l} x_{i_1} \cdot \cdots \cdot x_{i_l} \otimes_{\mathbb{Z}/p\mathbb{Z}} y_{j_1} \wedge \cdots \wedge y_{j_n} = 0$. But this implies that $kr^{j_1,\dots,j_n}_{i_1,\dots,i_l}$ is zero in $\mathbb{Z}/p\mathbb{Z}$ and therefore, all the elements $r^{j_1,\dots,j_n}_{i_1,\dots,i_l}$ are zero in $\mathbb{Z}/p\mathbb{Z}$. So a is the zero element. □

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