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Melnikov method for parabolic orbits

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Abstract. The present work completes the study of the conditions under which Melnikov method can be used when the unperturbed system has a parabolic periodic orbit with a homoclinic loop, by considering the case of orbits whose associated Poicaré map has linear part equal to the identity. The result is that the conditions for the persistence under perturbation of the invariant manifolds also ensure the convergence of the Melnikov integral and hence the applicability of the method.

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1 Motivation and statement of the problem

Melnikov method is a basic tool for the detection of horseshoes in perturbations of integrable systems. It was introduced by Melnikov, using ideas which go back to Poincaré, in [11] where a time periodic perturbation of a two dimensional system was considered. More recently, the method has been extended to perturbations of completely integrable Hamiltonian systems with two [13] and more [15] degrees of freedom, and to other classes of systems in arbitrary dimension, see for instance [9], [18]. Elementary proofs and standard applications of this method, such as the periodically forced Duffing oscillator, can be found in most textbooks of dynamical systems, for instance [16], [1]. One of the basic assumptions in this setting is that the unperturbed system possesses a hyperbolic fixed point or periodic orbit with a homoclinic loop.

During the 90's several papers have appeared, related to problems in Celestial Mechanics, where the method is applied to orbits which are not hyperbolic but parabolic, [17], [6], [4]. In fact, in the context of Celestial Mechanics, and more precisely of periodic orbits at infinity, there are several examples in which the unperturbed problem has a homoclinic loop doubly assymptotic to a periodic orbit of parabolic type, i.e., such that the associated Poincaré map has linear part equal to the identity. The most famous of this examples is the Sitnikov problem treated in [12], where it is shown that chaotic behaviour arises as a consequence of transversal intersection of the stable and unstable manifolds of the parabolic orbit at infinity for the perturbed system. Other examples are the restricted three-body problem and the collinear three-body problem, see [10]. The ubiquitous nature of parabolic orbits at infinity in problems in Celestial Mechanics motivated McGehee's famous paper [10] on the existence and regularity of invariant manifolds for degenerated fixed points of parabolic type.

The point we wish to make in this paper is that the applicability of Melnikov method, which does not automatically follow from McGehee's theorem, can be proved whenever the assumptions of the theorem hold.

The related and complementary case of parabolic orbits whose associated Poincaré map has linear part equal to the identity plus a nilpotent term, which shows up naturally in the context of mechanical systems with potentials with degenerated extrema, was treated in [2] and in [8] from the point of view of the existence of invariant manifolds, and in [3] for the applicability of Melnikov method. So, together with these three papers and with [10], the present work completes the study of the conditions under which the usual tools for the detection of chaotic behaviour can be used when the unperturbed system has a parabolic orbit with a homoclinic loop. The final conclusion validates all the applications that can be found in the literature because it turns out that, whenever we have persistence of the invariant manifolds, it is possible to apply Melnikov method.

The technical difficulty of extending Melnikov method to the parabolic case comes from the problem of convergence of the Melnikov function. In the remaining part of this section we shall recall the main steps in the derivation of the Melnikov function in the simplest setting of a periodically perturbed one degree of freedom Hamiltonian system, because our interest will be to addres this convergence problem. However the proof of convergence given in Section 2 carries over to other formulations of Melnikov method. For instance Theorem 2.4 can be applied to Robinson's formulation [14] of the method, which is particularly well suited to the planar restricted three-body problem, one of the degenerated problems to which McGehee's theorem [10] and the results of this paper apply.

Consider then a system of the form

$$\dot{\mathbf{q}} = JD_{\mathbf{q}}H(\mathbf{q}) + \varepsilon g(\mathbf{q}, t, \varepsilon) \tag{1.1}$$

with $H : U \to \mathbb{R}$ and $g : U \times \mathbb{R} \times B_{\varepsilon_0} \to \mathbb{R}^2$ where $U \subset \mathbb{R}^2$ is open, B_{ε_0} the ball of radius ε_0, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, H and g are analytic and g is T-periodic in t. We shall assume that system (1.1) possesses a fixed point at $\mathbf{q} = 0$, that this fixed point is parabolic with null linear part and that there exists a homoclinic connection γ asssociated to $\mathbf{q} = 0$ for the unperturbed system ($\varepsilon = 0$).

In these conditions, the time-T Poincaré map $F_\varepsilon : U \to \mathbb{R}^2$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, is analytic with respect to x, y, and ε, and of the form

$$F_\varepsilon(x, y) = (id + p + r)(x, y), \tag{1.2}$$

where $(x, y) = \mathbf{q}$, id is the identity, $p = (p_1, p_2)$ is a homogeneous polynomial of degree $n \geq 2$, and $r = (r_1, r_2)$ consists of terms of degree at least $n + 1$. If we impose further that for $x > 0$,

$$p_1(x, 0) < 0, \quad p_2(x, 0) = 0, \quad \frac{\partial p_2}{\partial y}(x, 0) > 0, \tag{1.3}$$

the map F_ε is in the conditions of ([10], Theorem 1) and so there exist $\beta, \delta > 0$ such that, considering the sector centered on the positive x-axis $B(\beta, \delta) = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \delta, \quad |y| \leq \beta x\}$, the local stable invariant manifold

$$W^s_{\delta,\varepsilon} = \left\{ (x, y) \in B(\beta, \delta) \;:\; F^k_\varepsilon(x, y) \in B(\beta, \delta),\, \forall k > 0, \text{ and } \lim_{k \to +\infty} F^k_\varepsilon(x, y) = 0 \right\}$$

is the graph of a differentiable function $\psi_\varepsilon : [0, \delta] \to \mathbb{R}$. Similar assertions are valid for the local unstable invariant manifold $W^u_{\delta,\varepsilon}$ by considering F_ε^{-1}.

To these local stable and unstable manifolds correspond local stable and unstable manifolds $\tilde{W}^{s,u}_{\delta,\varepsilon}$ of the parabolic periodic orbit at $q = 0$ of the autonomous system associated to (1.1),

$$\begin{aligned} \dot{\mathbf{q}} &= JD_{\mathbf{q}}H(\mathbf{q}) + \varepsilon\, g(\mathbf{q}, \theta, \varepsilon), \\ \dot{\theta} &= 1, \end{aligned} \tag{1.4}$$

where $(\mathbf{q},\theta) \in U \times S^1$. The homoclinic loop γ of the unperturbed system (1.1) becomes for system (1.4) when $\varepsilon = 0$ a two-dimensional manifold Γ of orbits doubly assymptotic to the periodic orbit at $\mathbf{q} = 0$. The local invariant manifolds of (1.4) can be extended by the flow to obtain $\tilde{W}_\varepsilon^{s,u}$. To detect the possible intersections of $\tilde{W}_\varepsilon^s$ and $\tilde{W}_\varepsilon^u$ consider a surface of section

$$\Sigma^{\theta_0} = \{(\mathbf{q},\theta) \in U \times S^1 \,:\, \theta = \theta_0\},$$

a homoclinic orbit $\mathbf{q}_0(t)$ of the unperturbed system, and $\mathbf{p} = (\mathbf{q}_0(-t_0),\theta_0)$, the intersection of this orbit with Σ^{θ_0}. On Σ^{θ_0}, let $L_\mathbf{p}$ be the straight line through $\mathbf{p}$ and orthogonal to Γ at $\mathbf{p}$. Denote by $p^u(\varepsilon)$ (resp. $p^s(\varepsilon)$) the intersection $L_\mathbf{p} \cap \tilde{W}_\varepsilon^u$ (resp. $L_\mathbf{p} \cap \tilde{W}_\varepsilon^s$). Then the signed distance $d(t_0,\theta_0,\varepsilon)$ between $\tilde{W}_\varepsilon^u$ and $\tilde{W}_\varepsilon^s$ at $\mathbf{p}$ is

$$d(t_0,\theta_0,\varepsilon) = (q^u(t_0,\theta_0,\varepsilon) - q^s(t_0,\theta_0,\varepsilon)) \cdot \frac{D_\mathbf{q} H\left(\mathbf{q}_0(-t_0)\right)}{\|D_\mathbf{q} H\left(\mathbf{q}_0(-t_0)\right)\|},$$

where the dot means the scalar product and $(q^{u,s}(t_0,\theta_0,\varepsilon),\theta_0) = p^{u,s}(\varepsilon)$.

The Melnikov function $M(t_0,\theta_0)$ is defined from the first terms of the Taylor expansion of $d(t_0,\theta_0,\varepsilon)$ around $\varepsilon = 0$,

$$\begin{aligned} M(t_0,\theta_0) &= \frac{\partial d(t_0,\theta_0,\varepsilon)}{\partial \varepsilon} \|D_\mathbf{q} H\left(\mathbf{q}_0(-t_0)\right)\| \\ &= D_\mathbf{q} H\left(\mathbf{q}_0(-t_0)\right) \cdot \left(\frac{\partial}{\partial\varepsilon} q^u(t_0,\theta_0,0) - \frac{\partial}{\partial\varepsilon} q^s(t_0,\theta_0,0)\right). \end{aligned}$$

The method consists in giving a computable expression for $M(t_0,\theta_0)$ in terms of the orbits of the unperturbed system. In order to do that, the time dependent Melnikov function $M(t;t_0,\theta_0)$ is defined as

$$M(t;t_0,\theta_0) = D_\mathbf{q} H\left(\mathbf{q}_0(t-t_0)\right) \cdot \left(\frac{\partial}{\partial\varepsilon} q^u(t;t_0,\theta_0,0) - \frac{\partial}{\partial\varepsilon} q^s(t;t_0,\theta_0,0)\right), \quad (1.5)$$

where $\frac{\partial}{\partial\varepsilon} q^{u,s}(t;t_0,\theta_0,0)$ are the solutions of the first order variational equations with initial conditions such that $M(t_0,\theta_0) = M(0;t_0,\theta_0)$. Using this property, the derivative with respect to t of (1.5) may be computed to obtain the following alternative expression for the Melnikov function,

$$\begin{aligned} M(t_0,\theta_0) = &\int_{-\tau}^{\tau} D_\mathbf{q} H\left(\mathbf{q}_0(t-t_0)\right) \cdot g\left(\mathbf{q}_0(t-t_0), t+\theta_0, 0\right) \, dt \\ &+ \left(\Delta^u(-\tau) - \Delta^s(\tau)\right), \end{aligned} \quad (1.6)$$

where we have used the notation

$$\Delta^{u,s}(t) = D_\mathbf{q} H\left(\mathbf{q}_0(t-t_0)\right) \cdot \frac{\partial}{\partial\varepsilon} q^{u,s}(t;t_0,\theta_0,0).$$

Suppose that we can assure that

$$\lim_{t\to+\infty} \Delta^s(t) = \lim_{t\to+\infty} \Delta^u(-t) = 0. \tag{1.7}$$

Then the Melnikov function is given by the convergent improper integral

$$M(t_0,\theta_0) = \int_{-\infty}^{+\infty} D_{\mathbf{q}}H\left(\mathbf{q}_0(t-t_0)\right)\cdot g\left(\mathbf{q}_0(t-t_0), t+\theta_0, 0\right)\, dt. \tag{1.8}$$

In the hyperbolic case, what makes the proof of this convergence work is the hyperbolic invariant manifold theory and in particular the contraction along the stable manifold, see [14] for a discusion. In the parabolic case we do not have asymptotic contraction, however it is still possible using the properties of the invariant manifolds to show that (1.7) still holds. This is the main result of this paper, Theorem 2.1.

Once (1.7) is established, the usual consequences follow: an isolated (with respect to the t_0-coordinate) zero $(\tilde{t_0},\tilde{\theta_0})$ of (1.8) implies the transversal intersection of $\tilde{W}^s_\varepsilon$ and $\tilde{W}^u_\varepsilon$; moreover the condition $\frac{\partial M}{\partial t_0}(\tilde{t_0},\tilde{\theta_0}) \neq 0$ may be checked by diferentiating under the integral sign in (1.8).

The fact that, both in the restricted three body problem and in Sitnikov's problem, the parabolic orbit at infinity is in the conditions of Theorem 2.1 folllows immediately from the computations of the associated Poincaré map done in [10]. Therefore, Theorem 2.1 justifies the applicability of Melnikov's method as employed in [17] and in [6]. In Section 3, we consider yet another example, the Gyldén problem, a time dependent perturbation of the Kepler problem which has several applications in celestial mechanics and in cosmology.

We finish by remarking that, as argued in [3], the general case when the period T of the perturbation g in (1.1) depends on ε may be reduced, performing the change of time scale $t' = \frac{T(0)}{T(\varepsilon)}\, t$, to the case considered in this paper.

2 Dynamics on the stable manifolds and proof of the main result

In this section we shall prove the necessary estimates to obtain Theorem 2.1, which states that Equations (1.7) do indeed hold under the conditions of McGehee's theorem of existence of invariant manifolds; therefore their intersections may be detected by means of the Melnikov's function (1.8).

We shall use the following notation for the family F_ε of analytic maps (1.2),

$$F_\varepsilon(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \sum_{i+j=n} a_{ij}(\varepsilon)x^iy^j \\ \sum_{i+j=n} b_{ij}(\varepsilon)x^iy^j \end{pmatrix} + O_{n+1}(x,y), \tag{2.1}$$

and we shall assume conditions (1.3), which become

$$a_{n\,0}(\varepsilon) < 0\,, \quad b_{n\,0}(\varepsilon) = 0\,, \quad b_{n-1\,1}(\varepsilon) > 0. \tag{2.2}$$

Under these conditions, the quadratic approximation of F_ε has a stable manifold which coincides with the positive x-axis and McGehee's theorem ensures that the full system has a local stable manifold $W^s_{\delta,\varepsilon}$ which is the graph of a differentiable function $\psi_\varepsilon : [0, \delta) \to \mathbb{R}$. In the following lemmas we shall give estimates for the rate of contraction and for the rate of convergence to the origin along this stable manifold. Similar estimates can be given for the unstable manifold using the map F_ε^{-1}.

We shall use the following properties of the function ψ_ε on $(0, \delta)$:

(i) $\psi_\varepsilon(x) = O(x^2)$

(ii) $\psi'_\varepsilon(x) = O(x)$

(iii) $|\psi_\varepsilon(x) - \psi_0(x)| \le \varepsilon C x^2$,

where C is a constant. The basic facts behind these three properties are contained in [10] and they have been used before in [3] with the purpose of proving analogous properties. We shall not repeat here the details but we shall refer to these papers for a sketch of the proof. Property (i) may be obtained following the same reasoning of [10], Section 4, using quadratic estimates instead of the estimates of [10], Proposition 7, just as in [3], Lemma 3.1. Property (ii) follows from (i) as in [3], Lemma 3.2. Finally, (iii) is a consequence of (i) and the analytic dependence on ε of the invariant manifolds, as in [3], Lemma 3.4.

Lemma 2.1 *If $\delta > 0$ is small enough and $\mathbf{q}_1 = (x_1, y_1)$, $\mathbf{q}_2 = (x_2, y_2) \in W^s_{\delta,0}$ then*

$$\|F_0(\mathbf{q}_2) - F_0(\mathbf{q}_1)\| \le \left(1 + \frac{1}{3} a_{n\,0} \bar{x}^{n-1}\right) \|\mathbf{q}_2 - \mathbf{q}_1\|$$

where $\bar{x} = \max\{x_1, x_2\}$.

Proof. Assume $x_1 < x_2$. We have

$$F_0(\mathbf{q}_2) - F_0(\mathbf{q}_1) = \int_0^1 DF_0\,(s(t))\,(\mathbf{q}_2 - \mathbf{q}_1)\,dt,$$

with $s(t) = \mathbf{q}_1 + t(\mathbf{q}_2 - \mathbf{q}_1)$, $t \in [0, 1]$, and

$$DF_0(x, y) = \begin{pmatrix} 1 + \sum_{i+j=n} i a_{ij} x^{i-1} y^j & \sum_{i+j=n} j a_{ij} x^i y^{j-1} \\ \sum_{i+j=n} i b_{ij} x^{i-1} y^j & 1 + \sum_{i+j=n} j b_{ij} x^i y^{j-1} \end{pmatrix} + O_n(x, y),$$

where $a_{ij} = a_{ij}(0)$ and $b_{ij} = b_{ij}(0)$. Denote by

$$\begin{aligned} A_1(x, y) &= 1 + \sum_{i+j=n} i a_{ij} x^{i-1} y^j, \quad A_2(x, y) = \sum_{i+j=n} j a_{ij} x^i y^{j-1}, \\ B_1(x, y) &= \sum_{i+j=n} i b_{ij} x^{i-1} y^j, \quad B_2(x, y) = 1 + \sum_{i+j=n} j b_{ij} x^i y^{j-1}. \end{aligned}$$

Then, using that by property (i), $\psi_0(x) = O(x^2)$,

$$\begin{aligned}
&\|F_0(\mathbf{q}_2) - F_0(\mathbf{q}_1)\|^2 \\
&= \left((x_2 - x_1) \left(1 + \int_0^1 A_1(s(t))\, dt \right) \right. \\
&\quad \left. + (\psi_0(x_2) - \psi_0(x_1)) \int_0^1 A_2(s(t))\, dt \right)^2 \\
&\quad + \left((x_2 - x_1) \int_0^1 B_1(s(t))\, dt + (\psi_0(x_2) \right. \\
&\quad \left. - \psi_0(x_1)) \left(1 + \int_0^1 B_2(s(t))\, dt \right) \right)^2 \\
&\le (x_2 - x_1)^2 (1 + 2a_{n\,0} x_2^{n-1} + O(x_2^n)) \\
&\quad + (\psi_0(x_2) - \psi_0(x_1))^2\, O(x_2^{2n-2}) \\
&\quad + 2(x_2 - x_1)(\psi_0(x_2) - \psi_0(x_1))\, O(x_2^{n-1}) \\
&\quad + (\psi_0(x_2) - \psi_0(x_1))^2 (1 + 2b_{n-1\,1} x_2^{n-1}) \\
&= (1 + a_{n\,0} x_2^{n-1} + O(x_2^n))((x_2 - x_1)^2 + (\psi_0(x_2) - \psi_0(x_1))^2) \\
&\quad + a_{n\,0} x_2^{n-1} (x_2 - x_1)^2 + (2b_{n-1\,1} - a_{n\,0}) x_2^{n-1} (\psi_0(x_2) - \psi_0(x_1))^2 \\
&\quad + 2(x_2 - x_1)(\psi_0(x_2) - \psi_0(x_1))\, O(x_2^{n-1}).
\end{aligned}$$

The above inequality may be rewritten for $\delta > 0$ small enough as

$$\|F_0(\mathbf{q}_2) - F_0(\mathbf{q}_1)\|^2 \le \left(1 + \frac{8}{9} a_{n\,0} x_2^{n-1} \right) \|\mathbf{q}_2 - \mathbf{q}_1\|^2 , \tag{2.3}$$

because the remaining terms in the right-hand side are bounded from above by zero. In fact,

$$a_{n\,0} + (2b_{n-1\,1} - a_{n\,0}) \left(\frac{\psi_0(x_2) - \psi_0(x_1)}{x_2 - x_1} \right)^2 + 2 \left(\frac{\psi_0(x_2) - \psi_0(x_1)}{x_2 - x_1} \right) O(1)$$

has the sign of $a_{n\,0}$ for $\delta > 0$ small enough taking into account that, by property (ii), $\psi_0'(x) = O(x)$.

The result follows from inequality (2.3). □

Lemma 2.2 *For $(x, y) \in W^s_{\delta,\varepsilon}$ and $\delta > 0$ sufficiently small, if*

$$x < (a(n-1)k + c)^{-1/(n-1)}$$

then for all $m \in \mathbb{N}$

$$\pi_1 F_\varepsilon^m (x, y) < (a(n-1)(k + m) + c)^{-1/(n-1)} ,$$

where $\pi_1(x, y) = x$, $a = \frac{8}{9} |a_{n\,0}(\varepsilon)|$, $k \in \mathbb{N}$ is big enough and $c \in \mathbb{R}$ is an arbitrary constant.

Proof. Using induction it is enough to show that for $k \in \mathbb{N}$ big enough and

$$\bar{x} < (a(n-1)k + c)^{-1/(n-1)}$$

we have

$$\pi_1 F_\varepsilon (\bar{x}, \psi_\varepsilon(\bar{x})) < (a(n-1)(k+1) + c)^{-1/(n-1)} .$$

From (2.1), (2.2) and using that by property (i), $y = \psi_\varepsilon(x) = O(x^2)$, we get

$$\begin{aligned} \pi_1 F_\varepsilon(x,y) &= x + \sum_{i+j=n} a_{ij}(\varepsilon) x^i y^j + O_{n+1}(x,y) \\ &= x + a_{n\,0}(\varepsilon)x^n + \sum_{i+j=n,\, j\geq 1} a_{ij} x^i y^j + O_{n+1}(x,y) \\ &< x - \frac{8}{9}\,|a_{n\,0}(\varepsilon)|\, x^n = x - a x^n. \end{aligned}$$

Then

$$\pi_1 F_\varepsilon (\bar{x}, \psi_\varepsilon(\bar{x})) < \left(\frac{1}{a(n-1)k+c}\right)^{1/(n-1)} - a \left(\frac{1}{a(n-1)k+c}\right)^{n/(n-1)} .$$

We want to show that

$$\left(\frac{1}{a(n-1)k+c}\right)^{\frac{1}{n-1}} - a \left(\frac{1}{a(n-1)k+c}\right)^{\frac{n}{n-1}} \leq \left(\frac{1}{a(n-1)(k+1)+c}\right)^{\frac{1}{n-1}} ,$$

or equivalently,

$$1 - \frac{1}{(n-1)k + \frac{c}{a}} \leq \left(\frac{1}{1 + \frac{1}{k + \frac{c}{a(n-1)}}}\right)^{1/(n-1)} . \tag{2.4}$$

Letting $z = 1/(k + c/[a(n-1)])$, (2.4) may be written as

$$(1+z)^{-1/(n-1)} \geq 1 - \frac{z}{n-1}.$$

Hence, (2.4) follows for $k \in \mathbb{N}$ big enough from the convexity of $(1+z)^{-1/(n-1)}$ on $z \in [0, +\infty)$. □

Lemma 2.3 *There exist $\delta > 0$ and positive constants C_1 and C_2 such that if $\mathbf{q}_1 \in W^s_{\delta,\,0}$ and $\mathbf{q}_2 \in W^s_{\delta,\,\varepsilon}$, with*

$$\|\mathbf{q}_1 - \mathbf{q}_2\| < C_1\, \varepsilon,$$

then

$$\|F^m_\varepsilon(\mathbf{q}_2) - F^m_0(\mathbf{q}_1)\| \leq \varepsilon \left(C_1 + C_2 \sum_{j=0}^{m-1} \frac{1}{((n-1)j + c)^{2/(n-1)}} \right) , \ \forall\, m \in \mathbb{N}.$$

Proof. For a given $m \in \mathbb{N}$, denote by

$$\Delta_m = \|F_\varepsilon^m(\mathbf{q}_2) - F_0^m(\mathbf{q}_1)\|,$$

and define

$$\omega_m = (\pi_1 F_\varepsilon^m(\mathbf{q}_2), \psi_0\,(\pi_1 F_\varepsilon^m(\mathbf{q}_2))) \in W^s_{\delta,\,0}.$$

Then,

$$\begin{aligned}\Delta_{m+1} \leq\ & \|F_\varepsilon F_\varepsilon^m(\mathbf{q}_2) - F_0 F_\varepsilon^m(\mathbf{q}_2)\| + \|F_0 F_\varepsilon^m(\mathbf{q}_2) - F_0(\omega_m)\| \\ & + \|F_0(\omega_m) - F_0 F_0^m(\mathbf{q}_1)\|\,. \end{aligned} \tag{2.5}$$

Because of the form of F_ε,

$$\|F_\varepsilon(\mathbf{q}) - F_0(\mathbf{q})\| \leq \varepsilon\,\bar{C}_1\ \|\mathbf{q}\|^n \tag{2.6}$$

for some constant $\bar{C}_1$ in a neighbourhood of $\mathbf{q} = 0$. On the other hand, by property (iii), if $x \in (0, \delta]$ then

$$|\psi_\varepsilon(x) - \psi_0(x)| \leq \varepsilon\,\bar{C}_2\,x^2$$

for some constant $\bar{C}_2$ and this implies

$$\|F_\varepsilon^m(\mathbf{q}_2) - \omega_m\| \leq \varepsilon\,\bar{C}_2\ (\pi_1 F_\varepsilon^m(\mathbf{q}_2))^2\,. \tag{2.7}$$

Using (2.5), (2.6), (2.7), Lemma 2.1, and also the fact that if δ is small, $\|DF_0(x,y)\| \leq 2$, we have

$$\Delta_{m+1} \leq \varepsilon\,\bar{C}_1\ (\pi_1 F_\varepsilon^m(\mathbf{q}_2))^n + 2\varepsilon\bar{C}_2\,(\pi_1 F_\varepsilon^m(\mathbf{q}_2))^2 + \|\omega_m - F_0^m(\mathbf{q}_1)\|\,.$$

By the triangle inequality,

$$\|\omega_m - F_0^m(\mathbf{q}_1)\| \leq \|\omega_m - F_\varepsilon^m(\mathbf{q}_2)\| + \Delta_m,$$

and we get the recurrence relation

$$\begin{aligned}\Delta_{m+1} &\leq \varepsilon\bar{C}_1\,(\pi_1 F_\varepsilon^m(\mathbf{q}_2))^n + 3\varepsilon\bar{C}_2\,(\pi_1 F_\varepsilon^m(\mathbf{q}_2))^2 + \Delta_m \\ &= \varepsilon(3\bar{C}_2 + \bar{C}_1\,(\pi_1 F_\varepsilon^m(\mathbf{q}_2))^{n-2})\,(\pi_1 F_\varepsilon^m(\mathbf{q}_2))^2 + \Delta_m \\ &\leq \varepsilon\,C_2\,(\pi_1 F_\varepsilon^m(\mathbf{q}_2))^2 + \Delta_m,\end{aligned}$$

which, applied inductively, yields

$$\begin{aligned}\Delta_m &\leq \varepsilon\,C_2\,\sum_{j=0}^{m-1}\left(\pi_1 F_\varepsilon^j(\mathbf{q}_2)\right)^2 + \|\mathbf{q}_2 - \mathbf{q}_1\| \\ &< \varepsilon\,\left(C_1 + C_2\,\sum_{j=0}^{m-1}\left(\pi_1 F_\varepsilon^j(\mathbf{q}_2)\right)^2\right) \\ &\leq \varepsilon\,\left(C_1 + C_2\,\sum_{j=0}^{m-1}\frac{1}{((n-1)j + c)^{2/(n-1)}}\right)\,,\end{aligned}$$

where for the last inequality we have used Lemma 2.2. □

Theorem 2.1 *Suppose that there exists a rotation in the* (x, y)*-plane which brings the Poincaré map* $F_\varepsilon(1.2)$ *of system* (1.1) *to a form which verifies condition* (1.3), *and that the inverse* F_ε^{-1} *may be also brought by a suitable rotation to verify conditions* (1.3). *Then the following properties hold:*

(a) $\lim_{\tau\to+\infty} \Delta^s(\tau) = \lim_{\tau\to+\infty} \Delta^u(-\tau) = 0$
(b) *The integral* $\int_{-\infty}^{+\infty} D_{\mathbf{q}}H\left(\mathbf{q}_0(t-t_0)\right) \cdot g\left(\mathbf{q}_0(t-t_0), t+\theta_0, 0\right)\, dt$ *converges.*

Proof. As in the standard proof of Melnikov's method, (b) follows directly from (a) and the fact that the first member of (1.6) does not depend on τ.

In order to prove (a) we have

$$\|\Delta^{u,s}(t)\| \leq \|D_{\mathbf{q}}H\left(\mathbf{q}_0(t-t_0)\right)\| \left\| \frac{\partial q^{u,s}}{\partial \varepsilon}(t; t_0, \theta_0, 0) \right\| . \tag{2.8}$$

By Lemma 2.2 and the fact that the Hamiltonian vector field is of order n, the asymptotic behaviour as $t \to \pm\infty$ of the first factor in the right-hand side of (2.8) is of the form

$$\|D_{\mathbf{q}}H(\mathbf{q}_0(t-t_0))\| \approx \frac{k_1}{((n-1)\frac{t}{T} + k_2)^{n/(n-1)}} ,$$

with k_1, k_2 constants.

On the other hand, by Lemma 2.3, if we take an arbitrary sequence $t_m = \bar{t} \pm mT \to \pm\infty$ when $m \to +\infty$,

$$\left\| \frac{\partial q^{u,s}}{\partial \varepsilon}(t_m; t_0, \theta_0, 0) \right\| \leq C_1 + C_2 \sum_{j=0}^{m-1} \frac{1}{((n-1)j + c)^{2/(n-1)}} .$$

Hence,

$$\|\Delta^{u,s}(t_m)\| \leq \frac{k_1}{((n-1)m + k_2)^{n/(n-1)}} \left(C_1 + C_2 \sum_{j=0}^{m-1} \frac{1}{((n-1)j + c)^{2/(n-1)}} \right)$$

which tends to zero when $m \to +\infty$. □

3 Example

Consider the Gyldén problem, a time dependent perturbation of the Kepler problem, whose Hamiltonian is of the form

$$H(\mathbf{r}, \mathbf{p}, t) = \frac{p^2}{2} - \frac{1}{r} + \epsilon\, G(r, t), \tag{3.1}$$

where $r = \|\mathbf{r}\|$, $p = \|\mathbf{p}\|$ and $G(r,t)$ vanishes at infinity and is 2π periodic in t. This problem was considered in [5] and in [7], where sufficient conditions are given for the existence of transversal homoclinic points and hence of chaotic behaviour for the perturbed system. While the methods in [5] are self-contained, the aproach in [7], based on the change to McGehee's variables and on the usual geometric construction of the Melnikov function, requires justification as given by Theorem 2.1.

Since the perturbation does not break the symmetry of the Kepler problem, the equations of motion for the perturbed system are

$$\begin{aligned} \dot{r} &= y \\ \dot{y} &= -\frac{1}{r^2} + \frac{c^2}{r^3} - \epsilon\, G_r(r,t), \end{aligned} \tag{3.2}$$

where c is the constant angular momentum and G_r the derivative of G with respect to r.

In terms of McGehee's variables $(x,y) = (r^{-1/2}, y)$, (3.2) becomes

$$\begin{aligned} \dot{x} &= x^3 y/2 \\ \dot{y} &= -x^4 + c^2 x^6 - \epsilon g(x,t), \end{aligned} \tag{3.3}$$

where $g(x,t) = G_r(x^{-2}, t)$. Assume that $g(x,t)$ is analytic in x and of the form $g_3(t)x^3 + g_4(t)x^4 + h.o.t.$, where $g_3(t)$ and $g_4(t)$ have zero mean. With respect to the perturbing term in the original Hamiltonian (3.1), this ammounts to assume that $G(r,t)$ is analytic in $r^{-1/2}$ and that the terms of order $r^{-1/2}$ and r^{-1} have zero mean. Then, the time 2π Poincaré map around the parabolic periodic orbit at infinity $(x=0, y=0)$ is given by

$$F_\epsilon(x,y) = (x - \pi x^3 y + r_1(x,y,\epsilon), y - 2\pi x^4 + r_2(x,y,\epsilon)), \tag{3.4}$$

where $r_i(x,y,\epsilon)$, $i = 1,2$, are of order at least five.

Let us now check that (3.4) is in the conditions of Theorem 2.1. Changing variables according to $x = (u-v)/\sqrt{2}$, $y = u+v$, the Poincaré map becomes

$$(u,v) \to \Big(u - \frac{\pi}{2}u(u-v)^3 + \tilde{r}_1(u,v,\epsilon), v + \frac{\pi}{2}v(u-v)^3 + \tilde{r}_2(u,v,\epsilon)\Big), \tag{3.5}$$

where $\tilde{r}_i(u,v,\epsilon)$, $i=1,2$, are of order at least five, which clearly verifies condition (1.3). Similarly,

$$F_\epsilon^{-1}(x,y) = (x + \pi x^3 y + q_1(x,y,\epsilon), y + 2\pi x^4 q_2(x,y,\epsilon), \tag{3.6}$$

becomes under the change of variables $x = (u+v)/\sqrt{2}$, $y = -u+v$

$$(u,v) \to \Big(u - \frac{\pi}{2}u(u+v)^3 + \tilde{q}_1(u,v,\epsilon), v + \frac{\pi}{2}v(u+v)^3 + \tilde{q}_2(u,v,\epsilon)\Big), \tag{3.7}$$

which again verifies condition (1.3).

Therefore, by Theorem 2.1, the isolated zeroes, in the t_0 coordinate, of the Melnikov function defined by the convergent integral (1.8) correspond to intersections of the stable and unstable manifolds of the parabolic orbit at infinity of the perturbed system (3.1). In this case, (1.8) is

$$M(t_0, \theta_0) = -\int_{-\infty}^{+\infty} y_0(t - t_0) g(x_0(t - t_0), t + \theta_0)\ dt,$$

where $(x_0(t - t_0), y_0(t - t_0))$ is a homoclinic solution of the unperturbed system (3.3) which can be easily computed using the energy integral. Choosing, as in [7], $G(r, t) = \epsilon\mu(t)/r$, with $\mu(t)$ a zero mean 2π periodic function, we have

$$M(t_0, 0) = -\int_{-\infty}^{+\infty} y_0(\tau) x_0^4(\tau) \mu(\tau + t_0)\ d\tau,$$

which is also a zero mean periodic function and hence has in general at least two zeroes.

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