Comment on "A Collection That is Not a Set!?"

(Lisboa: OFA-5, Feb. 2009)

Eduardo Castro*

1

In the first part of his essay, Gonçalo Santos makes a distinction between two types of aggregates: fusions and sets (or collections). Fusions are entities that are nothing besides the sum of its parts; sets (or collections) are entities that are *something more* than the sum of its parts. This definition of sets (or collections) seems unsatisfactory. What is the mean of the expression "something more", really? According to the metaphor view, the expression "something more" is referent to a box that contains all the set's elements. So, this view claims that sets are formed by its elements and a box that contains all such elements. Synthetically,

Sets = elements + box

Gonçalo Santos rejects this view, because it does not make sense to think that every set (or collection) of objects implies the existence of an abstract actual box that contains the objects of those sets.

Next, Gonçalo Santos links the definition's problem of sets to an epistemic challenge about the knowledge of mathematical entities, in particular, the knowledge of sets. He argues that someone that is committed with the existence of sets must also explain how we can get knowledge of them, since this explanation is not clear at all: entities in space-time are not causal connected with abstract entities outside of space-time, so it appears that it is impossible get knowledge of abstract entities, such as sets.

Some strategies can be used in order to respond to this epistemic challenge. I will summarise three (here, I follow Balaguer (2008)):

1. Rejecting the view that human mind is purely physical.

First, Plato argued that all knowledge is nothing more than remembrance. Since before we born our immaterial souls lived in the world of forms, then "mathematical learning is really just a process of coming to remember what we knew before we born".

---

* Dep. Mathematics, UBI and LanCog, CFUL. I am grateful to Teresa Marques for the invitation to take a comment in this workshop.

Second, and more recently, Kurt Gödel defended the existence of faculty of mathematical intuition, similar to the faculty of sense perception that allows acquiring knowledge about abstract objects, such as mathematical objects. Needless to say that these two views are completely out of fashion, nowadays.

2. Rejecting the thesis that mathematical objects exist outside of space-time.

In "Mathematical Realism", Penelope Maddy defended that when we looks to a box with three eggs we see three eggs and, in the same place of the eggs, we see a set of three eggs. *Prima facie*, this view is highly vulnerable as the above views of Gödel or Plato. However, is fair to notice that Maddy's view is thoroughly supported by arguments from studies of sense perception in the childhood.

3. Defending that all knowledge is acquired by empirical confirmation, including knowledge about abstract objects.

The most famous argument about this third strategy is the indispensability argument of Quine-Putnam that could be stated along the following lines.

(1) Mathematical entities, such as sets, are indispensable to our best scientific theories.

(2) Our best scientific theories are empirical tested as a whole, including much mathematics. (This is, confirmational holism.)

(3) We are committed with all and only those entities that are indispensable to our best scientific theories. That is, we are committed with those entities that are quantified in our best scientific theories. (This is, Quine's criterion of ontological commitment.)

(∴) We have empirical evidence for the commitment with mathematical entities (in particular, with sets).

(Other possible strategies concerning 3 are Structuralism (Resnik), Full-Blooded Platonism (Balaguer) and Necessity (Katz).)

2

In the second and third part of the essay, Gonçalo Santos is concerned with Russell's paradox and with the notions of proper classes and indefinite extensibility.

In order to block the emergence of paradoxes, such as Russell paradox, in set theory the construction of sets is built by levels. This hierarchical structure implies that the Russell's paradox is nothing more than a proof of the following theorem:

Theorem: *there is no universal set; that is, there is no set that contains all sets.*
(The proof of this theorem, as Gonçalo Santos said, is an absurd reduction proof.)
Proof:

Let us suppose that:

There is a set U that contains all sets.

Then, from the Axiom of Comprehension, we can form the set $Ru = \{x \in U: x \notin x\}$

Now we ask: is $Ru \in Ru$ (is Ru member of itself)?

Let the set x be the following particular set $x = Ru$,

From definition of Ru, it follows that $Ru \in U: Ru \notin Ru$. Absurd.

Then it is false that there is a set U that contains all sets, that is *there is no set that contains all sets*.♦

However, sets theorists talk about collections, in particular, in ZFC we have two types of collections (or extensions):

- Collections of sets that are sets, such as the infinite set $V\omega$, (referred as classes).

- Collections of sets that are not sets, such as the collection U above (referred as proper classes).

The point of Gonçalo Santos it is that this distinction between two types of collections is artificial because nothing on the proof above implies this distinction. Then, and alternatively, he follows Dummett making an intuitionist distinction between concepts with fix extensions and concepts indefinitely extensible.

Summing up:

- In set theory we have two types of collections (or extensions): 1) collections (or extensions) that are sets; 2) collections (or extensions) that are not sets (proper classes).

- In intuitionism we have two types of concepts: 1) concepts with fixed extensions; 2) concepts that could be indefinitely extensible.

In the beginning of the twentieth century, these two conceptions emerged in philosophy of mathematics as a dispute between realism and anti-realism. For realists the mathematical universe is independent of our minds and language; humans only aspire to discover mathematical properties in that universe; for the anti-realists the mathematical universe is dependent of our minds and language and, so, is a construction of humans by valid logic reasoning.

The realism vs. anti-realism dispute is not a only a dispute about the metaphysical status of mathematical reality. This dispute goes deep as logic goes. Intuitionists claim that the law of excluded middle (*tertium non datur*) is not valid and that, in general, as a consequence from this invalidity, the rule of double negation elimination (an absurd reduction proof that infers A from $\sim\sim A$) is an invalid rule, also.

Gonçalo Santos rightly affirms that in the proof above there is not an appeal to the law of excluded middle, "so by it self the adoption of intuitionist logic would not help with dealing with the paradox". However, in this proof there is an appeal to an absurd reasoning – an absurd reduction proof. So, it seems that it could be interesting to explain if this is an intuitionist valid proof or not. That is, in this proof is there an appeal to the double negation elimination rule? Or, in fact, this proof just appeals to the converse of the double negation elimination rule?

My last criticism to the essay is concerned with Dummett's claim that "we should adopt intuitionist logic in mathematical reasoning". If intuitionist logic is the only valid logic, then classical mathematics, founded in classical logic, is full of invalid proofs, as any undergraduate book of mathematics could show. This epistemic normative shift from logic to mathematics is not so undisputable as it seems.

Contemporary, some extreme views of naturalism defend that epistemic normative claims about science or mathematics are anti-naturalistic claims. According to these views, the role of philosophy, in particular, the role of philosophy of science, broadly speaking, should be a merely description of scientific beliefs or methods. For example, the doctrine of Mathematical Naturalism's of Penelope Maddy defends that practice mathematics is the last court for philosophy. In particular, she argues that philosophy occupies no privilege position from which to critique the practice of natural science or mathematics and "if our philosophical account of mathematics comes into

conflict with successful mathematical practice, it is philosophy that must give". (Maddy (1998), p. 161). When we look to mathematics we see that the overwhelming majority of mathematics is practiced according with the principles of classical mathematics (that is, the principles of bivalence and excluded middle and the double negation elimination rule are taken as valid), so the philosophical advice of intuitionism is ill founded. There is no reason for mathematicians change their methods in order to accommodate philosophical criticism, as Dummett advocates.

One the other hand, some, like D. C. McCarty (2005), claim that contemporary intuitionism is no more a philosophy of mathematics, but a school of mathematics *per se* independent of classical mathematics. They also argue that it is a very powerful and successful branch, despite the few mathematicians doing this kind of job. No area of modern mathematics has proven being resistant to intuitionist approaches, even subject matters of physics, such as quantum mechanics. Moreover, there are also subjects in intuitionism, such as the Kleen realizability structure for set theory, that are closed to classical mathematics. They claim that David Hilbert was wrong when we wrote "Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists". Alternatively, they argue that the principle of excluded middle, mathematically, is no principle at all, but a false generality, and a proof is given (founded on the Brouwer work). For them, depriving the intuitionist of the principle of excluded middle "is not like depriving the astronomer of telescope, but like depriving the aviator of a diving bell" (McCarty (2005), p. 383).

If this is minority of mathematicians is right, and there is much evidence that it is right, then contemporary naturalism, that advocates a mere rubber-stamp role to the philosopher, is historical refuted. Contemporary intuitionism is a well defined subject matter of mathematics that goes back to Brouwer but, and more importantly, was vigorous prompted by the philosophy of Heyting and Dummett.

References:

Balaguer, M. (2008), "Mathematical Platonism", in Gold & Simons (eds.), *Proof and other Dilemas*, (USA: Mathematical Association of America Publications, 2008), p. 179-204.

Maddy, P. (1998), *Naturalism in Mathematics*, (GB: OUP).

McCarty, D. C. (2005), "Intuitionism in Mathematics", in Shapiro (ed.), *The Oxford Handbook of Philosophy of Mathematics and Logic*, (NY: OUP, 2005), p. 356-386.